Stationary point process, Palm measure and collision risk

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Abstract. The classical probability of collision between an aircraft whose the path crosses a flow of aircraft is derived under the assumption that it is described by a Poisson process. Using the so-called Palm measure, we extend the classical result to a stationary point process.

Keywords. Palm measure, point process, collision risk

Introduction

Traditionally, when studying the simplest model of horizontal collision problem for two intersecting air-routes, the hypothesis of a Poisson stationary process is made. We show in this paper that the Poisson hypothesis is not necessary, at least at a first approximation. For that, we use the Palm measure of the stationary process. In the second Section, we recall the main notions about stationary point process, this allows us to introduce in the third Section the Palm measure. This part follows closely [1]. Finally in the last Section, we apply the previous results in the context of collision risk.

1. Point process

In many applications, we observe some discrete events occurring at times T_0, $T_1,\dots,T_n$; more formally these observations can be encompassed in the sequence $(T_n, n \in \mathbb{Z})$, where the $T_n \le T_{n+1}$ are random variables. We call this process a *point process* on $\mathbb{R}^+$, and we say that the point process is *simple* if $T_n < T_{n+1}$ for each n. Another point of view consists in counting the number $N((a,b])$ of events observed during the time interval $(a,b]$; then $N((a,b]) = \sum_{n\in\mathbb{Z}} 1_{(a,b]}(T_n)$ and $T_n = \inf\{t : N(-\infty,t]) = n\}$. We say that the point process is *stationary* if the joint probability of the number of events in m disjoint intervals $I_1,\cdots,I_m$ is invariant by translation, i.e. for all $m \in \mathbb{N}$ and all $t \in \mathbb{R}$

$$\mathbb{P}(N(I_1) = k_1,\cdots,N(I_m) = k_m) = \mathbb{P}(N(I_1+t) = k_1,\cdots,N(I_m+t) = k_m).$$

Let introduce for each t, the shift mapping θ_t defined on the probability space by $N(\theta_t\omega, I) = N(\omega, I+t)$. Therefore, the point process should be stationary if and only if

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$\mathbb{P} \circ \theta_t = \mathbb{P}$. These mappings satisfy the following properties: $\theta_t \circ \theta_s = \theta_{t+s}$, $\theta_t^{-1} = \theta_{-t}$ and for each $T_n \leq t < T_{n+1}$, $\theta_t T_n = T_0$ and $\theta_{T_{n+1}} = T_1$ with the convention that $T_0 \leq 0$ (see Figure 1).

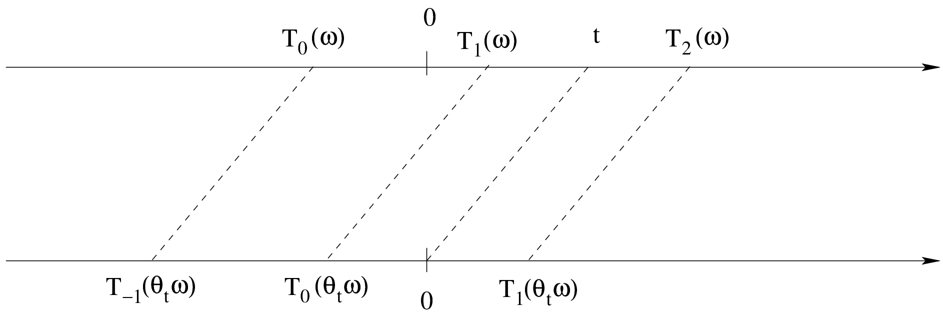

Figure 1. The shift. Note the numbering of the points T_i

In the following, we will consider only stationary point processes. The positive number $\lambda = \mathbb{E}(N((0,1]))$ is called the *intensity* of the point process; this number can be infinite, so we assume now that $\lambda > 0$ and is finite. Since the stationarity, for all interval I, the measure $\lambda(I) := \mathbb{E}(N(I))$ is invariant by translation, so it is proportional to the Lebesgue measure l on $\mathbb{R}$ and we deduce that $\lambda(I) = \lambda \cdot l(I)$, with a constant $\lambda > 0$.

2. Palm measure

An important area in which point processes are used is queueing theory. For example, the number of clients at a bank counter is a point process and the study of the counting process N is useful to decide to open another counter or not. Nevertheless, the point of view of the client is different, since the more pertinent for him is the waiting time at the instant he arrives in the queue or more generally all informations related to the counting process given its arrival time.

This suggests us to introduce a new probability P_N^0, called Palm measure, defined only on the event $T_0 = 0$, i.e. $P_N^0(T_0 = 0) = 1$. Due to the stationarity of the point process, the random variables $S_n = T_{n+1} - T_n$ are identically distributed. Therefore, on the event $\{T_0 = 0\}$, we could define the empirical function $F_0(t)$ of $T_1 = S_0$ and we will have $F_0(t) = P_N^0(S_n \leq t)$ for all n. Thus, the Palm measure need to be a measure P_N^0 such that $P_N^0(A) = 0$ for all measurable sets A such $A \cap \{T_0 = 0\} = \emptyset$. For that, we need to count the number of events in A, not by observing all the process but by shifting successively all the T_n at the origin of the time (that means the client moves to each T_n), and so we consider the sum $\sum_{n\in\mathbb{Z}} 1_A \circ \theta_{T_n}$. Nevertheless, the Palm measure being a probability we have to normalize the sum, from which the probability

$$P_N^0(A) = \frac{1}{\lambda l(C)} \mathbb{E}\left[(1_A \circ \theta_{T_n}) 1_C(T_n)\right] := \frac{1}{\lambda l(C)} \mathbb{E}\left[\int_C (1_A \circ \theta_s) N(ds)\right],$$

where C is any measurable subset of $\mathbb{R}$. Nevertheless, it can be proved that this probability is independent of C and is supported by the event $\{T_0 = 0\}$, so we adopt the following definition

$$P_N^0(A) = \frac{1}{\lambda}\mathbb{E}\left[(1_A \circ \theta_{T_n})1_{(0,1]}(T_n)\right].$$

Knowing the Palm probability, it is also possible to obtain the law of the Point process by the following relation (Inversion Formula of Ryll-Nardzewski and Slivnyak)

$$\mathbb{P}(A) = \lambda \int_0^\infty P_N^0(T_1 > t, \theta_t \in A)dt. \tag{1}$$

Let us come back to the empirical function $F_0(t)$ of the S_n under the probability P_N^0, i.e.

$$F_0(t) = P_N^0(S_n \leq t)$$

and introduce the *residual waiting time* at time $t \geq 0$ defined by

$$W(t) = T_{N(t)+1} - t$$

and the *spent waiting time* at time t

$$A(t) = t - T_{N(t)}.$$

For a stationary point process, you can take $t = 0$, in that case $W(0) = T_1$ and $A(0) = -T_0$. Using (1) we get for $A = \{T_1 > v, -T_0 > u\}$ with $u, v \geq 0$,

$$\mathbb{P}(T_1 > v, -T_0 > u) = \lambda \int_{v+u}^\infty (1 - F_0(s))ds,$$

from which we derive

$$\mathbb{P}(T_1 > v) = \lambda \int_v^\infty (1 - F_0(s))ds, \qquad \mathbb{P}(-T_0 > u) = \lambda \int_u^\infty (1 - F_0(s))ds.$$

We conclude that $-T_0$ and T_1 are identically distributed for the probability P. Moreover, for any $t \in [0, T_1]$, we have $S_0(\theta_t) = S_0$, from which we can derive the following identity available for any function f:

$$\mathbb{E}[f(S_0)] = \lambda \mathbb{E}_N^0[S_0 f(S_0)].$$

We then obtain directly the law of S_0 under $\mathbb{P}$:

$$\mathbb{P}(S_0 \leq x) = \int_{[0,x]} \lambda y F_0(dy),$$

which is in general different from $F_0(x)$. For instance,

$$\mathbb{E}(S_0) = \int_0^\infty \lambda y^2 F_0(dy) = \mathbb{E}_N^0(S_0)\left(1 + \frac{\mathrm{Var}_N^0(S_0)}{(\mathbb{E}_N^0(S_0))^2}\right),$$

since $\mathbb{E}_N^0(S_0) = 1/\lambda$, so $\mathbb{E}(S_0) = \mathbb{E}_N^0(S_0)$ iff under the Palm probability, the variance of S_0 is zero, therefore iff S_0 is a constant and thus also all the S_n.

3. Application to the collision risk

We consider two aircraft A_1 and A_2, each flying on straight-line paths at constant velocity $\mathbf{v_1}$ and $\mathbf{v_2}$ respectively. If the aircraft A_1 is considered fixed, the aircraft A_2 can be considered having a velocity $\mathbf{v}_r = \mathbf{v}_2 - \mathbf{v}_1$ relative to the aircraft A_1 and a relative position $\mathbf{r}(t)$ at time t given by, $\mathbf{r}(t) = \mathbf{r}(0) + \mathbf{v}_r t$.

A conflict is declared when the predicted position of the two aircraft are such that both horizontal and vertical separation parameters are infringed. Therefore, we associate a conflict volume to the aircraft A_1 which is an airspace formed around aircraft A_1 into which, if aircraft A_2 enters, a conflict is signalled. When this volume is a sphere a radius R, a conflict occurs when the predictive distance at the Closest Point of Approach (CPA) is smaller than or equal to R. The CPA is determined by the condition $\frac{d}{dt}(\mathbf{r}(t)\cdot\mathbf{r}(t)) = 0$ or

$$\mathbf{r}(t)\cdot\mathbf{v}_r = 0,$$

that means the relative distance $\mathbf{r}$ is orthogonal to the relative velocity $\mathbf{v}_r$. The time t_{CPA} of the CPA and the distance d_{CPA} at CPA are given respectively by

$$t_{\mathrm{CPA}} = -\frac{\mathbf{r}(0)\cdot\mathbf{v}_r}{\mathbf{v}_r\cdot\mathbf{v}_r}, \qquad d^2_{\mathrm{CPA}} = \mathbf{r}(0)\cdot\mathbf{r}(0) - \frac{(\mathbf{r}(0)\cdot\mathbf{v}_r)^2}{\mathbf{v}_r\cdot\mathbf{v}_r}.$$

Let $\theta_r \in [0, 2\pi[$ be the angle between $\mathbf{v}_r$ and $\mathbf{r}(0)$, i.e. $\mathbf{r}(0)\cdot\mathbf{v}_r = r(0)v_r\cos\theta_r$, with $r(0) = \|\mathbf{r}(0)\|$ and $v_r = \|\mathbf{v}_r\|$. If $\theta_r = 0$ then the aircraft A_2 is passing from the aircraft A_1 and this situation is not relevant, so by now we assume that $\theta_r \neq 0$. Introducing the distance $a = R/(|\sin\theta_r|)$, a conflict will occur if $r(0) \leq a$.

We consider now an aircraft A_2 whose the straight-line path crosses a flow of aircraft each of which are flying on straight-line path too. Moreover, we assume that aircraft A_1 of the flow are distributed according to a stationary point process with intensity λ. Let choose the time zero as the moment the aircraft A_2 crosses the flow and let T_0 and T_1 the time separations between the two closest aircraft A_1 of the flow to the aircraft A_2. That means, at time $t = 0$, an aircraft A_1 will arrive at the crossing point at time $-T_0$ and the other crossed this same point T_1 time units ago. If F_0 is the distributed function of T_1 under the Palm measure of the point process, the probability P_c to have a conflict between the aircraft A_2 and one of aircraft A_1 is given by

$$P_c = 1 - \mathbb{P}(-T_0 > a/v_1, T_1 > a/v_1) = 1 - \lambda\int_{2a/v_1}^{\infty}(1 - F_0(u))du = \lambda\int_0^{2a/v_1}(1 - F_0(u))du,$$

since $\int_0^\infty (1 - F_0(u))du = \mathbb{E}_N^0(T_1) = 1/\lambda$.

The quantity $2a/v_1$ being very small, we obtain the first order approximation $P_c \approx (2\lambda a)/v_1$ and if in addition, F_0 has a density f_0, we obtain a second order approximation

$$P_c \approx \frac{2\lambda a}{v_1}\left(1 - \frac{a}{v_1}f_0(0)\right).$$

3.1. *Poisson Process*

The Poisson process corresponds to the case $F_0(t) = 1 - e^{-\lambda t}$; so

$$P_c = 1 - e^{-2\lambda a/v_1} \approx \frac{2\lambda a}{v_1}\left(1 - \frac{\lambda a}{v_1}\right).$$

As $v_r \sin\theta_r = v_2 \sin\theta$ with θ the crossing angle between the two trajectories, we obtain the well-known formula

$$P_c \approx \frac{2\lambda a}{v_1}.$$

3.2. *Gamma Law*

Now, we assume that F_0 is a Gamma Law with parameters $p > 0$ and $\theta > 0$ whose the density is given by

$$f_0(u) = \frac{\theta^p}{\Gamma(p)} e^{-\theta u} u^{p-1} \qquad u \geq 0.$$

The expectation of this law being p/θ, we have to set $\lambda = \theta/p$, therefore

$$\begin{aligned}
P_c &= \frac{2\lambda a}{v_1} - \lambda \int_0^{2a/v_1} \int_0^t \frac{\theta^p}{\Gamma(p)} e^{-\theta u} u^{p-1} du dt \\
&= \frac{2\lambda a}{v_1} - \lambda \int_0^{2a/v_1} \frac{\theta^p}{\Gamma(p)} e^{-\theta u} u^{p-1} \int_u^{2a/v_1} dt du \\
&= \frac{2\lambda a}{v_1}\left[1 - \left(\frac{2a\theta}{v_1}\right)^p \frac{1}{\Gamma(p)} \int_0^1 (1-v) v^{p-1} e^{-\frac{2a\theta}{v_1} v} dv\right] \\
&= \frac{2\lambda a}{v_1}\left[1 - \left(\frac{2a\theta}{v_1}\right)^p \sum_{n=0}^{\infty} (-1)^n \left(\frac{\theta 2a}{v_1}\right)^n \frac{B(p+n,2)}{n!\Gamma(p)}\right] \\
&= \frac{2\lambda a}{v_1}\left[1 - \sum_{n=0}^{\infty} (-1)^n \left(\frac{2\theta a}{v_1}\right)^{n+p} \frac{B(p+n,2)}{n!\Gamma(p)}\right].
\end{aligned}$$

where $B(x,y) = \int_0^1 (1-v)^{y-1} v^{x-1} dv = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$. Thus the second order approximation is given by

$$P_c \approx \frac{2\lambda a}{v_1}\left[1 - \left(\frac{2\lambda a p}{v_1}\right)^p \frac{1}{\Gamma(p+2)}\right].$$

3.3. *Regular events with random translatories*

Here, we consider events which are regularly spaced out but with a possibly random translatory. So, the i-th event occurs at time

$$t_i = a_0 + ic + b_i, \qquad i = \dots, -1, 0, 1, \dots,$$

where c is spacing between the events in absence of random perturbations and the b_i are independent and identically distributed random variables whose the common empirical function, under the Palm measure, will be denoted by F_B. Then the i-th event is planned at time $a_0 + ic$, but its real time is moved by b_i. This type of process has been studied by T. Lewis and Govier [2,3] about the arrivals of tankers at a terminal.

We choose the event with indice $i = 0$ as initial time; this event being planned at time $-b_0$ with regard to the origin of time ($a_0 + b_0 = 0$), we deduce that, given b_0, the i-th event occurs at time $ic + b_i - b_0$, so

$$\mathbb{P}_N^0(t_i \le t | b_0) = F_B(t + b_0 - ic) - F_B(b_0 - ic).$$

It is important to note that the times t_i are not the times T_i of the point process induced by the events considered, since the j-th event can occur before the i-th event, even if $i < j$.

To obtain $F_0(t)$, it is enough to observe that $P_N^0(T_1 > t) = P_N^0(N(0,t] = 0)$, that means no event, except the 0-th event, has occurred during the time interval $(0,t]$. Thus

$$\mathbb{P}_N^0(T_1 > t | b_0) = \prod_{i=-\infty}^{\infty}{}' \left[1 - (F_B(t + b_0 - ic) - F_B(b_0 - ic))\right],$$

$$1 - F_0(t) = \int_{-\infty}^{\infty} \prod_{i=-\infty}^{\infty}{}' \left[1 - (F_B(t + b_0 - ic) - F_B(b_0 - ic))\right] F_B(db_0),$$

where $\prod_{i=-\infty}^{\prime\infty}$ denotes the product over all the indices i except $i = 0$.

Let assume now, that F_B has a density f_B with a finite value in zero. Then, for t small enough we get the following approximation

$$\begin{aligned}
F_0(t) &\approx 1 - \int_{-\infty}^{\infty} \prod_{i=-\infty}^{\infty}{}' \left[1 - t f_B(b_0 - ic)\right] f_B(b_0) db_0 \\
&\approx 1 - \int_{-\infty}^{\infty} \left[1 - t \sum_{i=-\infty}^{\infty}{}' f_B(b_0 - i)\right] f_B(b_0) db_0 \\
&= t \sum_{i=-\infty}^{\infty} \int_{-\infty}^{\infty} f_B(b_0 - i) f_B(b_0) db_0 - t \int_{-\infty}^{\infty} f_B^2(b_0) db_0 \\
&= t \sum_{i=-\infty}^{\infty} f_Z(-ic) - t f_Z(0),
\end{aligned}$$

where f_Z is the density of a random variable Z with same law that the difference of two random variables i.i.d. with density f_B.

The sum $c \sum f_Z(t - ic)$ has been investigated as an approximation for the integral $\int_{-\infty}^{\infty} f_Z(u) du$ by [5,4]; the difference is extremely small for all t, especially when $f_Z(x)$ is a normal p.d.f. We deduce the following new approximation

$$F_0(t) \approx t\left[c^{-1}\int_{-\infty}^{\infty} f_Z(u)du - f_Z(0)\right] = t\left(\frac{1}{c} - f_Z(0)\right).$$

Therefore, the probability P_c can be approximated by

$$P_c \approx \lambda \int_0^{2a/v_1}\left[1 - u(c^{-1} - f_Z(0))\right] du = \frac{2\lambda a}{v_1}\left[1 - \frac{a}{v_1}\left(\frac{1}{c} - f_Z(0)\right)\right].$$

For instance, if the B is distributed as a centred Gaussian variable with variance σ_b^2, then

$$f_Z(x) = \frac{1}{2\sigma_b \pi^{1/2}} \exp\left(-\frac{x^2}{4\sigma_b^2}\right),$$

and

$$P_c \approx \frac{2\lambda a}{v_1}\left[1 - \frac{a}{v_1}\left(\frac{1}{c} - \frac{1}{2\sigma_b \pi^{1/2}}\right)\right].$$

It remains to estimate the parameter λ, nevertheless, the process being stationary the mean number of events, during a period of t unit of time, is t/c, so λ is approximatively equal to $1/c$.

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