The *ex ante* $\alpha-$core for normal form games with uncertainty

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Abstract

In this paper we study the existence of the $\alpha-$core for an $n-$person game with incomplete information. We follow a Milgrom-Weber-Balder formulation of a game with incomplete information. The players adopt behavioral strategies represented by Young measures. The game unrolls in one step at the *ex ante* stage. In this context, the mixed-extensions of the utility functions are not quasi-concave, and as a result the classical Scarf's theorem cannot be applied. An approximation argument is used to overcome this lack of concavity.

Keywords: $\alpha-$core, Game with incomplete information, Normal Form Games, behavioral strategies, Game with uncertainty
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1. Introduction

The purpose of this work is to study a "cooperative" version ($\alpha-$core) of Milgrom-Webers's (Milgrom and Weber, 1985) model. We will work with a slightly different formulation developed in (Balder, 1988) using behavioral strategies. Concerning the α-core, let us say briefly that in the cooperative game theory players bargain to select a "collectively efficient" outcome. The concept of the core appears then as a key concept. It is intuitively defined as the set of payoff allocations (or decisions generating payoffs) relatively to which no coalition can make all its members better off. Its natural formulation for normal form games gives rise to the $\alpha-$core introduced by Aumann (1961).

The existence of the α-core was mainly proved by Scarf (1971) for normal form games with continuous quasi-concave payoffs and convex compact finite dimensional action spaces. Obviously, Scarf's result remains valid if players payoffs are only quasi-concave upper semi-continuous and the strategies spaces are convex compact subsets of arbitrary Hausdorff topological vector spaces. In a more general setting, Kajii (1992) proved the non

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vacuity of the α-core when players welfare is measured by means of non-ordered preferences. This answers the question of whether Scarf's result survives without transitivity and completeness of players preferences. In Kajii's framework, each player preference relation is described by a set-valued map associating to a given aggregate strategy a set of preferred strategies by the player. In this framework, Kajii proved an existence result for generalized games with open graph preferences and compact and convex action spaces. The compactness is assumed relatively to a topology derived from a norm which is a very restraining condition in an infinite dimension setting. This limitation has been overcome by Martins-da-Rocha and Yannelis (2011) using a Bewley-type limit argument (Bewley, 1978), a possible solution trick already discussed in Kajii (1992).

The concavity remained essential in these extensions. In the Milgrom-Webers's (Milgrom and Weber, 1985) model with behavioral strategies that we consider in this paper, the expected payoffs may not be quasi-concave even with some restricted class of underlying payoff functions. So a straightforward application of Scarf's method (or its generalizations) cannot be used. To be more explicit, as shown in Example 2 below, even if the utility function for each player is concave, its mixed-extension may fail to be quasi-concave. Scarf's proof works by proving that a canonical associated characteristic function form game is balanced by invoking the quasi-concavity of payoffs. Hence such argument does not apply in our context. One novelty of this paper is the recovery of a similar balancedness using an approximation argument: the density of the set of pure strategies in the set of behavioral ones which allows us to exploit the concavity of (the initial) payoffs.

For some insight into behavioral strategies (and their variants) in connection with the model that we deal with in this paper, let us begin by referring to the formulation of Milgrom and Weber (1985) of an *ad hoc* mathematical model for normal form games with incomplete information running in one step at the *ex ante* stage. This model relies on Harsanyi's work (Harsanyi, 1967–1968), assigning a set of types to players and a probability on this set intended to reflect the incomplete information aspect of the game.
In a closely related study, Radner and Rosenthal (1982) developed a model for Nash equilibrium with private information (each agent observes privately a realization of a random variable). They investigated finite games with finite action spaces and under the atomless property of the probabilities governing the agents' information which were in turn assumed to be independent. They established existence results of Nash equilibrium in pure strategy as a purification of equilibria in behavioral strategies. This technique is widespread in the literature, the reason being that in many situations, particularly dealing with Nash equilibrium (in incomplete information case), the existence results are obtained easily with behavioral/mixed strategies unlike in the case with pure strategies. This is also the method used by Schmeidler (1973) to prove the existence of pure strategy Nash equilibrium for games with a continuum of players. In their paper, Milgrom and Weber (1985) extended Radner and Rosenthal's results by proving the existence of Nash equilibrium in distributional strategies (non "disintegrated" behavioral strategies), and provided more general purification results. They established further approximate equilibria in pure

strategies. Note, however, that the action space remains finite for the existence of the pure strategy Nash equilibrium in Milgrom and Weber (1985).
Khan and Sun (1995) considered the model of Radner and Rosenthal. They generalized the existence results to games with countable action spaces. Khan and Sun (1995) worked out direct proofs of the existence of equilibria in pure strategies without passing through purification of behavioral strategies. This was achieved by developing mathematical tools on the distribution of correspondences, relying on a generalized version of the well known marriage lemma. Khan and Sun generalized, at the same time, to a countable action space, Schmeidlers' existence result of a pure strategy Nash equilibrium for games with a continuum of players. Khan *et al.* (1999) proved by a two-player example that for games with private information, the results of Radner and Rosenthal (1982), Khan and Sun (1995) and Milgrom and Weber (1985) cannot be extended to non countable action spaces.

Milgrom-Weber's model seems to us more appropriate for mathematical treatment. It was successfully reformulated by Balder (1988), using more elaborated and conventional mathematical tools (Young measures). Doing this, the known purification results (from behavioral strategies existence results) have been substantially improved in Balder (1988). Balder and Rustichini (1994) extended these results to games with an infinite set of players. Other interesting purification results can be found in Khan *et al.* (2006).

In this paper we adopt the formulation by Balder (1988) and we focus on *ex ante* core for normal form games without incentive compatibility constraints. This amounts to saying on one hand that the coalition formation is made at the *ex ante* stage. On the other hand, we assume, in line with the literature point of view (see for instance (Forges *et al.*, 2002)), that either the enforcement date is situated before the *interim* stage (there is no loss in randomness until the game takes place) or to require that all information become a public knowledge before the enforcement date. The studied model allows externalities in both types and actions. The *interim* core is interesting but far more complicated because of incentive problems.

The outline of the paper is as follows. In the next section we give the details of our game and introduce the corresponding equilibrium concept. We also recall some quick facts about Young measures which play a crucial role in our modelling. In Section 3, we establish the existence result. Then we end up in Section 4 with some remarks on our assumptions in the existence theorem.

2. The model and mathematical preliminaries

The cooperative game we consider in this paper takes place in the following framework. We have a finite set of players $N = \{1, ..., n\}$. Each player i observes an informational variable (or *type*) t_i whose values lie in some measurable space $(T_i, \mathcal{T}_i)$, where $\mathcal{T}_i$ is a σ-algebra. We abbreviate $T = \prod_{i \in N} T_i$ and endow T with the σ-algebra $\mathcal{T} = \bigotimes \mathcal{T}_i$. Let η be a probability on the product space $(T, \mathcal{T})$, which governs the random behavior of the information. The

marginally realized information type, for each player i, is governed by η_i, the marginal on $(T_i, \mathcal{T}_i)$ of η. For every $i \in N$, $\mathcal{T}_i$ is assumed to be η_i-complete. Associate to each player i an *action* space A_i which is a convex compact subset of a Banach space. We assume that the game takes place at the *ex ante* stage (see the introduction). That is each player i acts according to η_i.

We associate for each player i a utility function $U_i : T \times A \to \mathbb{R}$, where we abbreviate $A = \prod_{j \in N} A_j$. The payoff of the player i under the types vector t and the actions vector a is $U_i(t, a)$.

In this framework, *a pure strategy* for a player i is a measurable function $p_i : T_i \to A_i$. So, for every possible type $t_i \in T_i$, the player i associates an action $p_i(t_i)$. In this paper we are concerned with behavioral strategies. *A behavioral strategy* for a player i is a function from its type set to the set of probability measures $\mathcal{P}(A_i)$, *i.e.* $\delta_i : T_i \to \mathcal{P}(A_i), t_i \mapsto \delta_{it_i}$ satisfying : for every borel set $B \in \mathfrak{B}(A_i)$, the function $t_i \mapsto \delta_{it_i}(B)$ is $\mathcal{T}_i$-measurable. The interpretation of a behavioral strategy is that, under each possible type t_i, the player i selects an action in A_i according to the probability measure δ_{it_i}.

A behavioral strategy is nothing but a Young measure. It can be seen as a "measurable mixed strategy". Regarding the relatively weak regularity assumptions required for their use, Young measures seem adequate and very convenient in our framework. This will provides us with a wide powerful mathematical tools. Before pursuing, we briefly sketch some of their properties in the following subsection. For more details on the subject the reader is referred for e.g. to (Valadier, 1990; Castaing *et al.*, 2004; Balder, 2000).

2.1. Some facts about Young measures

Let $(\Omega, \mathcal{F}, \mu)$ be a finite measure space and X a compact metric topological space. Assume that $\mathcal{F}$ is μ-complete and endow X with its Borel σ-algebra $\mathfrak{B}(X)$. Denote by $\mathcal{R}(\Omega, X)$ the set of transition probabilities (or Young measures) with respect to Ω and X. That is the set of functions $\delta : \Omega \to \mathcal{P}(X)$, $\omega \mapsto \delta_\omega$ such that for every $B \in \mathfrak{B}(X)$, $\omega \mapsto \delta_\omega(B)$ is $\mathcal{F}$-measurable. This condition is equivalent to the measurability of the function $\delta : \Omega \to \mathcal{P}(X)$ where $\mathcal{P}(X)$ is endowed with the Borel $\sigma-$algebra generated by the weak (star) topology $\sigma(\mathcal{P}(X), C(X))$ (see Valadier (1990), Lemma A2 p. 178 and comment in p. 179, or Castaing *et al.* (2004), p. 19). It may be worth noting that we are speaking precisely about disintegrated Young measures. However, following a disintegration result (Valadier (1990), theorem A4, p. 157), the set $\mathcal{R}(\Omega, X)$ coincides (under our assumptions) with the set of measures defined on $\mathcal{F} \otimes \mathfrak{B}(X)$ whose projections on Ω equals μ. This more larger set, in a general situation, is referred to as the set of Young measures in (Castaing *et al.*, 2004; Valadier, 1990).

From generalized Fubini's theorem (Valadier (1990), theorem A1, p. 177), every Young measure $\delta \in \mathcal{R}(\Omega, X)$ induces a measure π_δ on the product space $(\Omega \times X, \mathcal{F} \otimes \mathfrak{B}(X))$ defined by

$$\forall A \in \mathcal{F}, B \in \mathfrak{B}(X),\ \pi_\delta(A \times B) = \int_A \delta_\omega(B) d\mu(\omega). \tag{1}$$

Furthermore, for all $\mathcal{F} \otimes \mathfrak{B}(X)$-measurable function $\Psi : \Omega \times X \to \overline{\mathbb{R}}$, which is π_δ-integrable,

we have

$$\int_{\Omega\times X}\Psi(\omega,x)d\pi_\delta=\int_\Omega\left[\int_X\Psi(\omega,x)d\delta_\omega(x)\right]d\mu(\omega). \tag{2}$$

The weak (star) topology (Balder (1988); Valadier (1990); Castaing *et al.* (2004)) on $\mathcal{R}(\Omega,X)$ is the coarsest topology making continuous the maps, $\delta\mapsto\int_{\Omega\times X}\Psi(\omega,x)d\pi_\delta$, where Ψ is a *Carathéodory integrand* (that is $\mathcal{F}\otimes\mathfrak{B}(X)$-measurable, continuous with respect to its second variable x and there exists a real μ-integrable function ϕ such that $|\Psi(\omega,x)|\leq\phi(\omega)$, for all $x\in X$). Endowed with this topology,

(R1) the space $\mathcal{R}(\Omega,X)$ is compact (see Castaing *et al.* (2004), theorem 4.3.5, p. 92, or Valadier (1990), theorem A4, p.179).

For any measurable function $f:\Omega\to X$, the corresponding degenerate (or Dirac) Young probability, denoted by $\epsilon(f)$, is defined for all $\omega\in\Omega$ by $\delta_\omega(f(\omega))=1$. Denote $\mathcal{D}(\Omega,X)$ the set of all such Young measures.

(R2) Assume that μ is non-atomic, then following Theorem 2.2.3, p. 40 in (Castaing *et al.*, 2004), the space $\mathcal{D}(\Omega,X)$ is dense in $\mathcal{R}(\Omega,X)$.

In our game context, this means that pure strategies are dense in behavioral strategies. For a degenerate Young measure $\delta=\epsilon(f)$, (2) reduces to

$$\int_{\Omega\times X}\Psi(\omega,x)d\pi_\delta=\int_\Omega\Psi(\omega,f(\omega))d\mu(\omega).$$

Let $(\Omega_i,\mathcal{F}_i,\mu_i)$, (resp. X_i), $i\in\{1,2\}$ be measure (resp. topological) spaces meeting previous requirements. Let μ be a measure defined on the product $\mathcal{F}_1\otimes\mathcal{F}_2$ such that its corresponding marginal on $(\Omega_i,\mathcal{F}_i)$ is μ_i. Consider the Young measure spaces $\mathcal{R}(\Omega_1,X_1)$ and $\mathcal{R}(\Omega_2,X_2)$. A product mapping $(\delta_1,\delta_2)\mapsto\delta_1\otimes\delta_2$ can be defined by setting $(\delta_1\otimes\delta_2)_{(\omega_1,\omega_2)}=\delta_{1_{\omega_1}}\otimes\delta_{2_{\omega_2}}$ and we have

(R3) Assume that μ is absolutely continuous with respect to $\mu_1\otimes\mu_2$, then the product mapping $(\delta_1,\delta_2)\mapsto\delta_1\otimes\delta_2$ from $\mathcal{R}(\Omega_1,X_1)\times\mathcal{R}(\Omega_2,X_2)$ into $\mathcal{R}(\Omega_1\times\Omega_2,X_1\times X_2)$ is continuous with respect to the weak topologies (see theorem 2.5 in Balder (1988)).

2.2. The α-core concept with behavioral strategies

Returning to the game, the set of strategies is $\prod_{j\in N}\mathcal{R}(T_j,A_j)$. When each player i plays his behavioral strategy $\delta_i:T_i\to\mathcal{P}(A_i)$, the expected payoff of i is $E_i:\prod_{j\in N}\mathcal{R}(T_j,A_j)\to\mathbb{R}$ defined by :

$$\begin{aligned}E_i(\delta_1,\delta_2,...,\delta_n)&=\int_T\left[\int_A U_i(t,a)d\delta_{1t_1}\otimes\delta_{2t_2}...\otimes\delta_{nt_n}(a)\right]d\eta(t)\\&=\int_T\left[\int_{A_1\times...\times A_n}U_i(t_1,...,t_n,a_1,...,a_n)d\delta_{1t_1}(a_1)d\delta_{2t_2}(a_2)...d\delta_{nt_n}(a_n)\right]d\eta(t)\end{aligned}$$

where we denoted $t = (t_1, ..., t_n)$ and $a = (a_1, ..., a_n)$.
In the sequel, we make the abbreviations $\mathcal{R} = \prod_{i \in N} \mathcal{R}(T_i, A_i)$ and for every coalition $S \subset N$, $\mathcal{R}_S = \prod_{i \in S} \mathcal{R}(T_i, A_i)$. The coalition $-S$ stands for $N \setminus S$. For $\delta_S \in \mathcal{R}_S$, $\delta_{-S} \in \mathcal{R}_{-S}$, we write (δ_S, δ_{-S}) for the element of $\mathcal{R}$ whose projections on $\mathcal{R}_S$ and $\mathcal{R}_{-S}$ are δ_S and δ_{-S} respectively. The spaces $\mathcal{R}(T_i, A_i)$, $i \in N$, are endowed with the weak topology described above.
Abbreviate and define the previous game by its main components :

$$G = (N, U_i, \mathcal{R}_{\{i\}}).$$

Thereafter, we introduce the adapted α-core equilibrium concept :

Definition 1. *A coalition $S \subset N$ blocks a behavioral strategy $\gamma = (\gamma_1, ..., \gamma_n) \in \mathcal{R}$ if and only if there exists $\delta_S \in \mathcal{R}_S$ such that for all $\delta_{-S} \in \mathcal{R}_{-S}$,*

$$E_i(\delta_S, \delta_{-S}) > E_i(\gamma), \forall i \in S$$

The $\alpha-$core of G is the set of behavioral strategies that are not blocked by any coalition.

In other words, a coalition blocks a given outcome of the game if it possesses a strategy making all its members better off regardless of the opponent coalition choices for strategy. The α-core describes situations in which no coalition has any incentive to form by playing a different strategy. Indeed, it cannot improve upon, relatively to the equilibrium strategy, the payoffs of all its members. In our model players' welfare is measured in terms of the expected utilities E_i computed from the common prior probability η and their respective state contingent utilities. Since the game takes place at the *ex ante* stage, we omit to consider communication games or any information sharing (see for instance (Myerson, 2007)). The following example illustrates the above *ex ante* blocking concept :

Example 1. Consider three players $N = \{1, 2, 3\}$. Set for every $i \in N$, $T_i = A_i = [0, 1]$. In the sequel λ refers to the Lebesgue measure on $[0, 1]$ and $\lambda_{|E}$ to its restriction to the measurable subset $E \subset [0, 1]$. Every T_i is endowed with its Borel $\sigma-$algebra on which we define the probability $\eta_i = \lambda$. Put $\eta = \eta_1 \otimes \eta_2 \otimes \eta_3$. The utility functions are defined for every $(t, a) \in T \times A$ by :

$$\begin{aligned} U_1(t, a) &= -t_2 t_3 (t_1 - \tfrac{1}{3}[a_1 + a_2 + a_3]), \\ U_2(t, a) &= -t_1 t_2 t_3 (\tfrac{1}{2} - \tfrac{1}{3}[a_1 + a_2 + a_3]), \\ U_3(t, a) &= -\tfrac{1}{3} t_3 (a_1^2 + a_1 + a_2 + a_3^2). \end{aligned}$$

Remark that the functions U_i are continuous, so the corresponding previous expectations E_i are well defined.
• Consider the behavioral strategy $\delta^1 = (\epsilon(0), \epsilon(0), \epsilon(0))$. That is, for every $i \in N$, δ_i^1 is the degenerate Young measure associated to the function $f_i \equiv 0$. Then,

$$E_1(\delta^1) = -\frac{1}{8}, \qquad E_2(\delta^1) = -\frac{1}{16} \qquad \text{and} \qquad E_3(\delta^1) = 0.$$

• The coalition $\{1,2\}$ blocks δ^1 by playing its behavioral strategy (δ_1^2, δ_2^2) defined by : $\delta_1^2 = \epsilon(g_1)$ and $\delta_2^2 = \epsilon(\frac{1}{2})$, where $\epsilon(g_1)$ is the Young measure associated to the function $g_1 : t_1 \mapsto t_1$ and $\epsilon(\frac{1}{2})$ is the Young measure associated to the constant function $g_2 \equiv \frac{1}{2}$. Indeed, for all $\delta_3 \in \mathcal{R}(T_3, A_3)$,

$$E_1(\delta_1^2, \delta_2^2, \delta_3) \quad = -\frac{1}{24} + \frac{1}{3}\int_T \left[\int_{A_3} t_2 t_3 a_3 d\delta_{3_{t_3}}(a_3)\right] d\eta.$$

It is clear that the minimum of E_1 over $\delta_3 \in \mathcal{R}(T_3, A_3)$ is reached at $\delta_3 = \epsilon(0)$ and $E_1(\delta_1^2, \delta_2^2, \epsilon(0)) = \frac{-1}{24} > E_1(\delta^1)$. Similarly, for all $\delta_3 \in \mathcal{R}(T_3, A_3)$, $E_2(\delta_1^2, \delta_2^2, \delta_3) \geq E_2(\delta_1^2, \delta_2^2, \epsilon(0)) = \frac{-1}{72} > E_2(\delta^1)$. For now, δ^1 is blocked by $\{1,2\}$ by playing $(\delta_1^2, \delta_2^2) = (\epsilon(g_1), \epsilon(\frac{1}{2}))$. To carry on this illustration, assume that the coalition $\{1,2\}$ plays $(\epsilon(g_1), \epsilon(\frac{1}{2}))$ and the player 3 plays his corresponding best strategy, which is obviously $\delta_3^2 = \epsilon(0)$. His expected gain is

$$E_3(\epsilon(g_1), \epsilon(\frac{1}{2}), \epsilon(0)) = -\frac{1}{3}\int_T t_3(t_1^2 + t_1 + \frac{1}{2})d\eta = -\frac{2}{9}$$

• Set $\delta^2 = (\epsilon(g_1), \epsilon(\frac{1}{2}), \epsilon(0))$. By a straightforward computation we can check that the grand coalition N blocks δ^2 by playing the behavioral strategy δ^3, defined by :

$$\delta_1^3 : t_1 \mapsto \frac{1}{t_1}\lambda_{|[0,t_1]}, \qquad \delta_2^3 = \frac{1}{2}\epsilon(\frac{1}{2}) + \frac{1}{2}\epsilon(0) \qquad \text{and} \qquad \delta_3^3 = \epsilon(\frac{3}{4}).$$

The corresponding expected payoffs are :

$$E_1(\delta_1^3, \delta_2^3, \delta_3^3) = -\frac{1}{48}, \qquad E_2(\delta_1^3, \delta_2^3, \delta_3^3) = -\frac{1}{144}, \qquad \text{and} \qquad E_3(\delta_1^3, \delta_2^3, \delta_3^3) = -\frac{169}{864}.$$

Note that, for every measurable $E \subset A_1$, the function $t_1 \mapsto \delta_{1_{t_1}}^3(E) = \frac{1}{t_1}\lambda([0,t_1] \cap E)$ is measurable as it is obviously continuous on $]0,1]$. This ensures that δ^3 is a Young measure.

3. Main result

Before stating the main result of this paper, let us expose our key assumptions. First we recall that

(A1) for every $i \in N$, the action space A_i is a convex compact subset of a Banach space

and assume that

(A2) for every $i \in N$, η_i is non-atomic and η is absolutely continuous with respect to $\bigotimes_{i=1}^{n} \eta_i$.

We also introduce the following assumptions on the payoff functions.

(A3) For every $i \in N$, U_i is measurable on the product $T \times A$, $U_i(t, \cdot)$ is continuous for every $t \in T$, and there exists an η-integrable real function $\phi : T \to \mathbb{R}$ such that $|U_i(t, a)| \leq \phi(t)$ for every $a \in A$ and a.e $t \in T$. That is to say, U_i is a Carathéodory integrand.

(A4) For every $i \in N$, $U_i(t, \cdot)$ is concave for a.e. $t \in T$.

Theorem 1. *Under assumptions (A1)–(A4), the $\alpha-$core of G is nonempty.*

The proof requires two preparatory lemmas.

Lemma 1. *E_i is continuous for every $i \in N$.*

Proof. By (R3) the map $(\gamma_1, \gamma_2, ..., \gamma_n) \mapsto \underset{j \in N}{\otimes} \gamma_j$ is continuous. Since for every $i \in N$, U_i is a Carathéodory integrand, the definition of the weak topology on $\mathcal{R}(T, A)$ provides *per se* the continuity of the real map $\delta \mapsto \int_{T \times A} U_i(t, a) d\pi_\delta$ defined on $\mathcal{R}(T, A)$. Hence, the averages $E_i, i \in N$, are continuous, as a composition of two continuous maps.

□

Lemma 2. *Let $S \subset N$ and $\varepsilon > 0$. Then, for every behavioral strategy profile $\delta_S = (\delta_{i_1}, ..., \delta_{i_{|S|}}) \in \mathcal{R}_S$ there exist pure strategies $f_i : T_i \to A_i$, $i \in S$, such that*

$$E_i(\epsilon(f_{i_1}), \epsilon(f_{i_2}), ..., \epsilon(f_{i_{|S|}}), \gamma_{-S}) \geq E_i(\delta_S, \gamma_{-S}) - \varepsilon, \forall i \in S, \forall \gamma_{-S} \in \mathcal{R}_{-S}.$$

Proof. From Lemma 1 and (R1), that is by continuity of the functions $E_i, i \in N$, and the compactness of $\mathcal{R}_{-S}$, the function

$$H : (\delta'_S, \delta_S) \mapsto \min_{i \in S} \min_{\gamma_{-S} \in \mathcal{R}_{-S}} [E_i(\delta'_S, \gamma_{-S}) - E_i(\delta_S, \gamma_{-S})]$$

is continuous on the product $\mathcal{R}_S \times \mathcal{R}_S$. Since $H(\delta_S, \delta_S) = 0$, there is a neighborhood $O(\delta_S)$ of δ_S such that,

$$H(\delta'_S, \delta_S) > -\varepsilon, \forall \delta'_S \in O(\delta_S).$$

By (R2), $\prod_{i \in S} \mathcal{D}(T_i, A_i)$ is dense in $\mathcal{R}_S$, so we can find measurable functions $f_i : T_i \to A_i$, $i \in S$, such that $\epsilon(f_S) := (\epsilon(f_{i_1}), ..., \epsilon(f_{i_{|S|}}))$ belongs to $O(\delta_S)$, that is

$$H(\epsilon(f_S), \delta_S) > -\varepsilon,$$

or equivalently

$$E_i(\epsilon(f_{i_1}), \epsilon(f_{i_2}), ..., \epsilon(f_{i_{|S|}}), \gamma_{-S}) > E_i(\delta_S, \gamma_{-S}) - \varepsilon, \forall i \in S, \forall \gamma_{-S} \in \mathcal{R}_{-S}.$$

□

Proof of Theorem 1. Scarf (1967) demonstrated a general core existence result for characteristic function form games. In order to use this result we must pass from the definition of the game in term of strategies and utility functions to the characteristic function form. Associate to the game G a characteristic function form game $G_C = (N, V)$, where $V : 2^N \to 2^{\mathbb{R}^N}$ is defined as follows :

$$V_S = \left\{ y \in \mathbb{R}^N : \begin{array}{l} \exists \delta_S \in \mathcal{R}_S, \forall \delta_{-S} \in \mathcal{R}_{-S}, \\ E_i(\delta_S, \delta_{-S}) \geq y_i, \forall i \in S. \end{array} \right\}.$$

A vector $y = (y_1, ..., y_n)$ is in V_S, if there is a behavioral strategy of members of S, which provides player i (for $i \in S$) with a utility of at least y_i for all strategy choices of the players not in S. A vector y is in the core of this game if $y \in V_N$ and y is not in the interior of V_S for any coalition S. It is obvious that to such element corresponds a behavioral strategy in the core of G. So, the goal now is to prove that G_C has a nonempty core. Following Scarf (1967), this will be true if we prove

(I) each V_S is closed and nonempty,

(II) each V_S is comprehensive (i.e., $y \in V_S$ and $x \leq y$ (in componentwise sense) implies $x \in V_S$),

(III) V_N is bounded from above,

(IV) G_C is balanced, that is to say, for every balanced collection of coalitions $\mathscr{C}$ with balancing weights $\alpha_S, S \in \mathscr{C}$, the following property holds :

$$\bigcap_{S \in \mathscr{C}} V_S \subset V_N,$$

where a collection of coalitions $\mathscr{C}$ is said to be balanced iff there is non negative weights $\alpha_S, S \in \mathscr{C}$, such that

$$\sum_{S \in \mathscr{C}, S \ni j} \alpha_S = 1, \forall j \in N.$$

The second item is trivial. The first item and the third item follow obviously from the continuity of the functions $E_i, i \in N$, and the compactness of the sets $\mathcal{R}, \mathcal{R}_S$ and $\mathcal{R}_{-S}$ for every $S \subset N$. Just remark, for the first item, that for every $S \subset N$, V_S is an upper section of a continuous function, precisely :

$$V_S = \left\{ y \in \mathbb{R}^n, \max_{\delta_S \in \mathcal{R}_S} \min_{\delta_{-S} \in \mathcal{R}_{-S}} \min_{i \in S} \{E_i(\delta_S, \delta_{-S}) - y_i\} \geq 0 \right\}$$

It remains to prove that G_C is balanced. Consider a balanced collection of coalitions $\mathscr{C}$ with the associated balancing weights $\alpha_S, S \in \mathscr{C}$, and let $y \in \bigcap_{S \in \mathscr{C}} V_S$. We must therefore demonstrate that $y \in V_N$. Since V_N is closed, it suffices to show that $(y_i - \varepsilon)_{i \in N} \in V_N$ for

every $\varepsilon > 0$. Indeed this will permit, thanks to Lemma 2, the use of pure strategies. From the definition of V, for every $S \in \mathscr{C}$ there exists $\delta_S \in \mathcal{R}_S$ such that for every $\gamma_{-S} \in \mathcal{R}_{-S}$, $E_j(\delta_S, \gamma_{-S}) \geq y_j$, for all $j \in S$. Let us given $\varepsilon > 0$. According to Lemma 2, for every $S \in \mathscr{C}$, there is measurable functions $f_i^S : T_i \to A_i, i \in S$, such that,

$$E_j((\epsilon(f_i^S))_{i\in S}, \gamma_{-S}) \geq E_j(\delta_S, \gamma_{-S}) - \varepsilon \geq y_j - \varepsilon, \forall \gamma_{-S} \in \mathcal{R}_{-S}, \forall j \in S. \tag{3}$$

In the sequel we apply a Scarf's trick (Scarf, 1971) in order to construct an element in $\mathcal{R}$ ensuring a utility of at least $y_j - \varepsilon$ for each player $j \in N$.
Define the function

$$f = (f_1, ..., f_n) = (\sum_{S\in\mathscr{C}, S\ni 1} \alpha_S f_1^S, ..., \sum_{S\in\mathscr{C}, S\ni n} \alpha_S f_n^S).$$

Fix an arbitrary index $j \in N$ and verify that f can be expressed as :

$$\sum_{S\in\mathscr{C}, S\ni j} \alpha_S (h_1^S, ..., h_n^S)$$

where $h_i^S : T_i \to A_i$ is defined by

$$h_i^S(t_i) = \begin{cases} f_i^S(t_i), & \text{if } i \in S, \\ \dfrac{\sum\limits_{E\ni i, E\not\ni j} \alpha_E f_i^E(t_i)}{\sum\limits_{E\ni i, E\not\ni j} \alpha_E}, & \text{if } i \notin S. \end{cases}$$

where the last summations (and all the following) are made, if not mentioned, over the coalitions E (or S) belonging to $\mathscr{C}$.
Indeed, for every i,

$$\sum_{S\in\mathscr{C}, S\ni j} \alpha_S h_i^S = \sum_{S\ni j, S\ni i} \alpha_S f_i^S + \sum_{S\ni j, S\not\ni i} \alpha_S \frac{\sum\limits_{E\ni i, E\not\ni j} \alpha_E f_i^E}{\sum\limits_{E\ni i, E\not\ni j} \alpha_E}.$$

To conclude that the previous quantity gives f_i, it suffices to remark that

$$\sum_{S\ni j, S\not\ni i} \alpha_S = \sum_{E\ni i, E\not\ni j} \alpha_E,$$

which is a consequence of the balancedness of the collection of coalitions :

$$1 = \sum_{S\ni j} \alpha_S = \sum_{S\ni j, S\not\ni i} \alpha_S + \sum_{S\ni i, S\ni j} \alpha_S = \sum_{E\ni i, E\not\ni j} \alpha_E + \sum_{S\ni i, S\ni j} \alpha_S = \sum_{E\ni i} \alpha_E = 1.$$

Now with the help of the concavity of U_j we obtain

$$\begin{aligned} E_j(\epsilon(f_1),...,\epsilon(f_n)) &= \int_T U_j(t_1,...,t_n,f_1(t_1),...,f_n(t_n))d\eta(t) \\ &= \int_T U_j\big(t_1,...,t_n,\sum_{S\ni j}\alpha_S(h_1^S(t_1),...,h_n^S(t_n))\big)d\eta(t) \\ &\geq \sum_{S\ni j}\alpha_S\int_T U_j(t_1,...,t_n,h_1^S(t_1),...,h_n^S(t_n))d\eta(t) \end{aligned}$$

But, for every $S \ni j$, that is every term in the right hand side of the last previous inequality, we can write,

$$\begin{aligned} \int_T U_j(t_1,...,t_n,h_1^S(t_1),...,h_n^S(t_n))d\eta(t) &= \int_T U_j(t_1,...,t_n,(f_i^S(t_i))_{i\in S},(h_i^S(t_i))_{i\in -S})d\eta(t) \\ &= E_j((\epsilon(f_i^S))_{i\in S},(\epsilon(h_i^S))_{i\in -S}) \\ &\geq y_j-\varepsilon \end{aligned}$$

where the last inequality comes from (3).
Hence,

$$E_j(\epsilon(f_1),...,\epsilon(f_n)) \geq y_j-\varepsilon$$

Since $j \in N$ is fixed arbitrarily, consider that the last inequality is true for all $j \in N$. So, we constructed an element $\delta=(\delta_1,...,\delta_n)=(\epsilon(f_1),...,\epsilon(f_n)) \in \mathcal{R}$ satisfying :

$$E_j(\delta) \geq y_j-\varepsilon, \forall j \in N.$$

□

4. Concluding remark

One can inquire whether the functions E_i defined above are quasi-concave on the space $\mathcal{R}$ under moderate conditions on the functions U_i (or under what conditions on U_i they are). Because, in such case, one can apply directly Scarf's (Scarf, 1971) (or a slightly modified version working in infinite dimension) existence results to prove the non-emptiness of the α-core of our game. This will make our results above superfluous. However, such an assertion is generally false. Indeed, the example below shows that under the concavity of U_i on $A=\prod A_i$, the functions E_i fail to be quasi-concave.

Example 2. Consider a game with n players, $n \geq 2$. For all $i \in \{1,2,3,...,n\}$, put $A_i=[0,1]$ and let (T_i,Σ_i) be a measurable space. Take η an arbitrary probability on $\prod_{i=1}^n T_i = T$, provided the previous requirements are satisfied, and denote η_i its marginal on T_i.
Let $\phi : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a linear map different from the 0 functional such that $(1,...,1) \in \ker\phi$. Take for instance $\phi(a_1,a_2,...,a_n)=\sum_{i=1}^n a_i - na_1$.

We emphasize that the purpose of this example is to show that even for concave utilities U_i, the expectations E_i may fail to be quasi-concave. This, in fact, will prove the non applicability of Scarf's non-emptiness result and will show accordingly the value added by our work. Henceforth, it suffices to construct a concave possible utility, denote it simply U, with a non quasi-concave corresponding expectation E.

Put $U : \prod_{i=1}^{n} A_i \longrightarrow \mathbb{R}$, $U(a_1, ..., a_n) = -|\phi(a_1, ..., a_n)|$. Then, U is obviously concave.

Let for all $i \in \{1, 2, 3, ..., n\}$,

- $f_i^1 : T_i \longrightarrow A_i$ the null function $f_i^1 \equiv 0$ and $\delta_i^1 = \epsilon(f_i^1)$ the associated Young measure. Put $\delta^1 = (\delta_1^1, \delta_2^1, \delta_3^1, ..., \delta_n^1)$.
- $f_i^2 : T_i \longrightarrow A_i$ the constant function $f_i^2 \equiv 1$ and $\delta_i^2 = \epsilon(f_i^2)$ the associated Young measure. Put $\delta^2 = (\delta_1^2, \delta_2^2, \delta_3^2, ..., \delta_n^2)$.

Then,

$$E(\delta^1) = \int_T U(f_1^1(t_1), f_2^1(t_2), f_3^1(t_3), ..., f_n^1(t_n))d\eta = U(0, ..., 0) = 0$$

and

$$E(\delta^2) = \int_T U(f_1^2(t_1), f_2^2(t_2), f_3^2(t_3), ..., f_n^2(t_n))d\eta = U(1, ..., 1) = 0.$$

But,

$$\begin{aligned}
E(\tfrac{\delta^1+\delta^2}{2}) &= \int_T \left[\int_A U(a) d \bigotimes_{i=1}^{n} \tfrac{\delta_{i_{t_i}}^1 + \delta_{i_{t_i}}^2}{2} \right] d\eta \\
&= \tfrac{1}{2^n} \int_T \left[\int_A U(a) d \bigotimes_{i=1}^{n} (\delta_{i_{t_i}}^1 + \delta_{i_{t_i}}^2) \right] d\eta \\
&= \tfrac{1}{2^n} \int_T \left[\int_{\prod_{i=1}^{n-1} A_i} \left(\int_{A_n} U(a) d(\delta_{n_{t_n}}^1 + \delta_{n_{t_n}}^2) \right) d \bigotimes_{i=1}^{n-1} (\delta_{i_{t_i}}^1 + \delta_{i_{t_i}}^2) \right] d\eta \\
&= \tfrac{1}{2^n} \int_T \left[\int_{\prod_{i=1}^{n-1} A_i} (U(a_1, ..., a_{n-1}, 0) + U(a_1, ..., a_{n-1}, 1)) d \bigotimes_{i=1}^{n-1} (\delta_{i_{t_i}}^1 + \delta_{i_{t_i}}^2) \right] d\eta \\
&= \tfrac{1}{2^n} \int_T \left[\int_{\prod_{i=1}^{n-2} A_i} \left(\int_{A_{n-1}} [U(a_1, ..., a_{n-1}, 0) + U(a_1, ..., a_{n-1}, 1)] \right. \right. \\
&\quad \left. \left. d(\delta_{(n-1)_{t_{n-1}}}^1 + \delta_{(n-1)_{t_{n-1}}}^2) \right) d \bigotimes_{i=1}^{n-2} (\delta_{i_{t_i}}^1 + \delta_{i_{t_i}}^2) \right] d\eta \\
E(\tfrac{\delta^1+\delta^2}{2}) &= \tfrac{1}{2^n} \int_T \left[\int_{\prod_{i=1}^{n-2} A_i} (U(a_1, ..., a_{n-2}, 0, 0) + U(a_1, ..., a_{n-2}, 1, 0) \right. \\
&\quad \left. + U(a_1, ..., a_{n-2}, 0, 1) + U(a_1, ..., a_{n-2}, 1, 1)) d \bigotimes_{i=1}^{n-2} (\delta_{i_{t_i}}^1 + \delta_{i_{t_i}}^2) \right] d\eta \\
&\quad \vdots \\
&= \tfrac{1}{2^n} \int_T \left[\sum_{S \subset \{1,2,3,...,n\}, S \neq \emptyset} U(\sum_{j \in S} e_j) \right] d\eta
\end{aligned}$$

where $e_j = (0, ..., 0, \underset{j^{\text{th}} \text{ component}}{1}, 0, ..., 0)$, for every $j \in \{1, ..., n\}$.
Since $\phi \not\equiv 0$, for at least one index $j \in \{1, 2, 3, ..., n\}$, $\phi(e_j) \neq 0$. Consequently $U(e_j) < 0$, hence,

$$\int_T \left[\sum_{S \subset \{1,2,3,...,n\}, S \neq \emptyset} U(\sum_{j \in S} e_j) \right] d\eta < 0.$$

That is,

$$E\left(\frac{\delta^1 + \delta^2}{2}\right) < \min\left\{E(\delta^1), E(\delta^2)\right\}.$$

Which means that E is not quasi-concave.

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