Range reduction using fixed points

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1 Introduction

Mixed Integer Nonlinear Programming (MINLP) problems of the form:

$$\left.\begin{array}{rrcl} \min & f(x) & & \\ \text{s.t.} & g(x) & \leq & 0 \\ & x & \in & X^0 \in \mathscr{I}^n \\ \forall i \in \mathcal{Z} & x_i & \in & \mathbb{Z} \end{array}\right\} \tag{1}$$

(where $f : \mathbb{R}^n \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}^m$ are continuous functions, $x \in \mathbb{R}^n$, $\mathscr{I}$ is the interval lattice, and $\mathcal{Z} \subseteq \mathcal{N} = \{1, \ldots, n\}$) are usually solved to optimality using the spatial Branch-and-Bound (sBB) algorithm [1]. The efficiency of sBB depends on many parameters, among which the width of the variable ranges at each node. The fastest range reduction algorithm is called Feasibility-Based Bounds Tightening (FBBT): it is an iterative procedure that propagates bounds up and down the expression trees [1] representing the constraints in (1), tightening them by using the constraint bounds $(-\infty, 0]$. Depending on the instance, and even limited to linear constraints only, FBBT may not converge finitely to its limit point. Tolerance-based termination criteria yield finite termination but, in general, in unbounded time (for every time bound, there is an instance exceeding it). So, although the FBBT is practically fast, its theoretical worst-case complexity status is far from satisfactory. We propose an alternative approach to FBBT based on fixed point equations $\mathcal{F}(X) = X$ for the variable bounding box, where $\mathcal{F}$ represents the action of the bounds propagation up and down the expression trees and $X \in \mathscr{I}^n$, and treat these equations as constraints of an auxiliary Mixed Integer Linear Program (MILP), which can be solved by a finitely terminating exponential time algorithm.

2 Semantic equations for the FBBT algorithm

To each variable x_j in (1) we associate an interval $X_j \subseteq X_j^0$. For each constraint $g_i(x) \leq 0$ we let $T_i = (V_i, A_i)$ be the directed expression tree [1] of g_i, and r_i its root node. Leaf nodes of T_i represent constants and variables, whereas other nodes represent operators: $(u, v) \in A_i$ only if v represents an argument of the operator represented by u in the function $g_i(x)$. For all $i \le m, v \in V_i$ we let $Y_{iv} \in \mathscr{I}$ be the interval associated to node v in T_i (initially set at $\top = (-\infty, \infty)$). For all $i \le m$ we denote the interval vector $(Y_{iu} \mid u$ is a leaf node of $V_i)$ by $\mathsf{varproj}(Y_i)$. For all $i \le m$ and leaf nodes $u \in V_i$ we can update Y_{iu} with $Y_{iu} \cap X_{\mathsf{index}(u)}$, where index maps variable leaf nodes to the corresponding variable indices; we denote the resulting interval vector by $\mathsf{restrict}(X, Y_i)$ for each $i \le m$.

For basic subtrees of T_i with arc sets $\{(\oplus, u), (\oplus, v)\}$, representing operations $x_u \oplus x_v$, $Y_{i,\oplus}$ is obtained by interval arithmetic on Y_{iu}, Y_{iv}. By recursion, this defines an operation $\mathsf{up}(T_i, Y_i)$ which updates $Y_i = (Y_{iv} \mid v \in V_i)$ starting from the leaf nodes. Supposing Y_{i,r_i} is not contained in $(-\infty, 0]$ (the constraint bound), then we can update Y_{i,r_i} with $Y_{i,r_i} \cap (-\infty, 0]$. Then, supposing the inverse operator to node r_i is well-defined (call it $\ominus$), for each $v \in \delta^+(r_i)$ we can propagate the update on Y_{i,r_i} to Y_{iv} using interval arithmetic on $\ominus(r_i, \delta^+(r_i) \smallsetminus \{v\})$. By recursion, this defines an operation $\mathsf{down}(T_i, Y_i)$ which updates Y_i starting from the root node. Since we perform up and down on each tree T_i in sequence, we update the ranges for T_{i-1} using those from T_i.

We let the operator $\mathcal{F}$ in the fixpoint equations represent an $\mathsf{up}/\mathsf{down}$ cycle carried out across $T = (T_1, \ldots, T_m)$. Letting $Y = (Y_1, \ldots, Y_m)$ and $Y_0 = Y_1$ we formally define:

$$\mathcal{F}(Y) = (\mathsf{down}(T_i, \mathsf{up}(T_i, \mathsf{restrict}(\mathsf{varproj}(Y_{i-1}), Y_i))) \mid i \le m). \tag{2}$$

Consider the sequence $\{Y^k\}$ s.t. $\forall k \in \mathbb{N}\ Y^k = \mathcal{F}(Y^{k-1})$ and $Y^0 = (\mathsf{up}(T_i, \mathsf{restrict}(X^0, Y_i)) \mid i \le m)$. Since $\mathcal{F}$ is monotone, by Tarski's Fixpoint Theorem [2] in the lattice $\mathscr{I}^{\bar{m}}$ (where $\bar{m} = \sum_{i \le m} |V_i|$), the sequence has a limit point which is the least fixed point of $\mathcal{F}$ (denoted by $\mathrm{lfp}(\mathcal{F})$). The same theorem also ensures that $\mathrm{lfp}(\mathcal{F})$ is the intersection of all the *post-fixpoints* of $\mathcal{F}$, i.e. of all those Y such that $Y \supseteq \mathcal{F}(Y)$. In other words, $\mathrm{lfp}(\mathcal{F})$ is given by the following problem:

$$\inf\{Y \mid Y \supseteq \mathcal{F}(Y)\}. \tag{3}$$

Since the natural interval width extended to vectors is monotonic with the natural order of the lattice $\mathscr{I}^{\bar{m}}$, (3) is equivalent to $(\min\{|Y| \mid Y \supseteq \mathcal{F}(Y)\})$. Next, we decompose the expression for $\mathcal{F}$ in (2) by introducing auxiliary interval vectors, and impose Y^0 as the first component of the sequence by requiring $Y_0 = Y_1 \cap Y_1^0$ with $Y_1^0 = \mathsf{up}(T_1, \mathsf{restrict}(X^0, Y_1))$. Thus, the solution to:

$$\left.\begin{array}{rrcl} & \min & & |Y| + |Y_0| + |X| + |\tilde{Y}| + |\bar{Y}| \\ & Y_0 & \supseteq & Y_1 \cap Y_1^0 \\ \forall i \le m & X & \supseteq & \mathsf{varproj}(Y_{i-1}) \\ \forall i \le m & \tilde{Y}_i & \supseteq & \mathsf{restrict}(X, Y_i) \\ \forall i \le m & \bar{Y}_i & \supseteq & \mathsf{up}(T_i, \tilde{Y}_i) \\ \forall i \le m & Y_i & \supseteq & \mathsf{down}(T_i, \bar{Y}_i) \end{array}\right\} \begin{array}{c} (a) \\ (b) \\ (c) \\ (d) \\ (e) \\ (f) \end{array} \tag{4}$$

is $\mathrm{lfp}(\mathcal{F})$, which is in turn the limit point of the FBBT. As long as the m constraints used in the FBBT are linear, (4) can be formulated as a MILP by introducing binary variables for the various interval operators appearing in (4).

References

[1] P. Belotti, J. Lee, L. Liberti, F. Margot, and A. Wächter. Branching and bounds tightening techniques for non-convex MINLP. *Optimization Methods and Software*, 24(4):597–634, 2009.

[2] A. Tarski. A lattice-theoretical fixpoint theorem and its applications. *Pacific Journal of Mathematics*, 5(2):285–309, 1955.