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To cite this version:
Alberto Costa, Sonia Cafieri, Pierre Hansen. Reformulation of a locally optimal heuristic for modularity maximization. ROADEF 2012, 13ème congrès annuel de la Société Française de Recherche Opérationnelle et d’Aide à la Décision, Apr 2012, Angers, France. <hal-00934798>

HAL Id: hal-00934798
https://hal-enac.archives-ouvertes.fr/hal-00934798
Submitted on 8 Apr 2014

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Reformulation of a locally optimal heuristic for modularity maximization

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Keywords: binary decomposition, clustering, modularity maximization, reformulation.

1 Introduction

A network, or graph, $G = (V, E)$ consists of a set of vertices $V = \{1, \ldots, n\}$ and a set of edges $E = \{1, \ldots, m\}$ connecting vertices. One of the most studied problems in the field of complex systems is to find communities, or clusters, in networks. A community consists of a subset $S$ of the vertices of $V$ where inner edges connecting pairs of vertices of $S$ are more dense than cut edges connecting vertices of $S$ to vertices of $V \setminus S$. Many criteria have been proposed to evaluate partitions of $V$ into communities. The best known of them appears to be the modularity, defined as follows by Newman and Girvan [9]:

$$Q = \sum_c Q_c = \sum_c \left( \frac{m_c}{m} - \frac{D_c^2}{4m^2} \right),$$

where $Q_c$ is the modularity of the cluster $c$, $m_c$ is the number of edges with both end vertices within the cluster $c$, $D_c$ is the sum of the degrees of the vertices in the cluster $c$, and $m$ is the number of edges of the whole network. The modularity is the difference between the fraction of edges within communities and the expected fraction of such edges in a random graph having the same distribution of degrees than the graph under study. In order to find a good partition into communities for a given network, according to Newman and Girvan one should maximize its modularity. This is a strongly NP-hard problem [3].

A few exact algorithms [1, 6, 10] and many heuristics have been proposed for network modularity maximization. They consist in divisive and agglomerative hierarchical clustering approaches [5, 8], as well as exact or approximate partitioning ones. In this paper, we focus on a recent locally optimal heuristic based on a hierarchical divisive approach [4]. We propose several ways to reformulate the model of [4] in order to accelerate the resolution by reducing efficiently the number of variables and constraints. Computation results are reported for a series of real-world problems from the literature in which the different reformulations are compared. It appears that computing times are very substantially reduced.

2 Initial model

The model used in the framework of the hierarchical divisive heuristic proposed in [4] to split a cluster $(V_c, E_c)$ into two clusters maximizing the modularity, and based on the one proposed in [10], is the following:
\[
\max \frac{1}{m} \left( m_1 + m_2 - \frac{1}{2m} \left( D_1^2 + \frac{D_c^2}{2} - D_1 D_c \right) \right) \tag{2}
\]

s.t.
\[
X_{i,j,1} \leq Y_i \quad \forall (v_i, v_j) \in E_c \tag{3}
\]
\[
X_{i,j,1} \leq Y_j \quad \forall (v_i, v_j) \in E_c \tag{4}
\]
\[
X_{i,j,2} \leq 1 - Y_i \quad \forall (v_i, v_j) \in E_c \tag{5}
\]
\[
X_{i,j,2} \leq 1 - Y_j \quad \forall (v_i, v_j) \in E_c \tag{6}
\]
\[
m_s = \sum_{(v_i,v_j) \in E_c} X_{i,j,s} \quad \forall s \in \{1,2\} \tag{7}
\]
\[
D_1 = \sum_{v_i \in V_c} k_i Y_{i,1} \tag{8}
\]
\[
Y_i \in \{0,1\} \quad \forall v_i \in V_c \tag{9}
\]
\[
X_{i,j,s} \geq 0 \quad \forall (v_i,v_j) \in E_c, \forall s \in \{1,2\}, \tag{10}
\]

where the variable \(X_{i,j,s}\) is equal to 1 if the edge \((v_i, v_j)\) is inside the community \(s\) (i.e., both vertices \(v_i\) and \(v_j\) are inside the community \(s\)) and 0 otherwise, \(Y_i\) is equal to 1 if the vertex \(v_i\) is inside the community 1, and 0 otherwise, and \(k_i\) is the degree of the vertex \(v_i\); note that \(D_c\) is a parameter, and it is known before solving the problem.

### 3 Reformulations

#### 3.1 Power of two reformulation

The heuristic proposed in [4] works by recursively splitting a cluster into two clusters in an optimal way (in the sense that the computed bipartition corresponds to the best possible modularity). The model is a quadratic integer programming one, with a convex relaxation. The only non-linear term is \(D_1^2\). The usual Branch-and-Bound approach implemented in CPLEX [7] is to relax the integrality constraints, solve the continuous quadratic program obtained and then branch. Alternately, one may linearize \(D_1^2\) by replacing it with its expansion in power of two, as proposed for mixed-integer quadratic programming in [2]:

\[
D_1 = \sum_{i=0}^{t} 2^i a_i, \quad a_i \in \{0,1\}. \tag{11}
\]

Therefore, the term \(D_1^2\) in (2) can be written as:

\[
D_1^2 = \sum_{l=0}^{t} 2^l a_l \cdot \sum_{h=0}^{l} 2^h a_h = \sum_{l=0}^{t} \sum_{h=0}^{l} 2^{l+h} a_l a_h = \sum_{l=0}^{t} \sum_{h=0}^{l} 2^{l+h} R_{lh} = \sum_{l=0}^{t} 2^{2l} a_l + \sum_{l=0}^{t} \sum_{h=0}^{l} 2^{l+h+1} R_{lh}, \tag{12}
\]

where \(R_{lh}\) is the linearization variable for \(a_l a_h\); hence, we have to adjoin the following constraints to our model:

\[
R_{lh} \geq a_l + a_h - 1, \quad \forall l \in \{0, \ldots, t\}, \forall h \in \{0, \ldots, l-1\}
\]
\[
R_{lh} \geq 0, \quad \forall l \in \{0, \ldots, t\}, \forall h \in \{0, \ldots, l-1\}.
\]

To estimate \(t\), recall that the maximum value which can be assumed by \(D_1\) is the sum of the degrees of all the vertices in the current cluster, that is \(D_c\). Moreover, from (11) the maximum possible value for \(D_1\) is \(2^{t+1} - 1\). Hence, \(t\) can be computed as:

\[
2^{t+1} - 1 \geq D_c \quad \Rightarrow \quad t = \lceil \log_2(D_c + 1) \rceil - 1. \tag{13}
\]
3.2 Change of variables

The model of [4] uses variables assigning edges or vertices to a specific community. When bipartitioning, as there are only two communities to be determined at each iteration, one can use other variables $S_{i,j}$, associated with the fact that the two end vertices $v_i$ and $v_j$ of an edge belong to the same cluster or not (i.e., $S_{i,j} = 1$ if $Y_i = Y_j$, and 0 otherwise). This leads to the following reformulation:

$$\max \frac{1}{m} \left( \sum_{(v_i, v_j) \in E_c} (2S_{i,j} - Y_i - Y_j) + \left| E_c \right| - \frac{1}{2m} \left( D_1^2 + \frac{D_c^2}{2} - D_1D_c \right) \right) \quad (14)$$

s.t. $S_{i,j} \leq Y_i \quad \forall (v_i, v_j) \in E_c$ \quad (15)
$s_i,j \leq Y_j \quad \forall (v_i, v_j) \in E_c$ \quad (16)
$D_1 = \sum_{v_i \in V_c} k_i Y_i$ \quad (17)
$Y_i \in \{0, 1\} \quad \forall v_i \in V_c$. \quad (18)

3.3 Symmetry breaking

To avoid considering twice equivalent solutions, one fixes a vertex to belong to the first (or second) community. It appears that the vertex with largest degree is a good choice.

4 Compact model

Applying all the reformulations presented in the previous sections leads to the following compact model:

$$\max \frac{1}{m} \left( \sum_{(v_i, v_j) \in E_c} (2S_{i,j} - Y_i - Y_j) + \left| E_c \right| - \frac{1}{2m} \left( \sum_{l=0}^{t} 2^{2l}a_l + \sum_{l=0}^{t} \sum_{h<l} 2^{l+h+1} R_{l,h} + \frac{D_c^2}{2} - D_1D_c \right) \right) \quad (19)$$

s.t. $S_{i,j} \leq Y_i \quad \forall (v_i, v_j) \in E_c$ \quad (20)
$s_i,j \leq Y_j \quad \forall (v_i, v_j) \in E_c$ \quad (21)
$R_{l,h} \geq a_l + a_h - 1 \quad \forall l \leq t, \forall h < l$ \quad (22)
$R_{l,h} \geq 0 \quad \forall l \leq t, \forall h < l$ \quad (23)
$D_1 = \sum_{l=0}^{t} 2^l a_l$ \quad (24)
$D_1 = \sum_{v_i \in V_c} k_i Y_i$ \quad (25)
$Y_g = 0, \quad g = \arg \max \{ k_i, \forall v_i \in V_c \}$ \quad (26)
$Y_i \in \{0, 1\} \quad \forall v_i \in V_c$ \quad (27)
$a_l \in \{0, 1\} \quad \forall l \leq t$. \quad (28)

This model has $|V_c| + t + 1$ binary variables, $|E_c| + \frac{t^2 + t}{2} + 1$ continuous variables and $2|E_c| + t^2 + t + 3$ constraints, while the initial model has $|V_c|$ binary variables, $2|E_c| + 3$ continuous variables and $6|E_c| + 3$ constraints.

5 Results

Table 1 presents the comparison of computing times for the initial model and the final one. Results have been obtained on a 2.4GHz Intel Xeon CPU of a computer with 24 GB RAM.
running Linux and CPLEX 12.2 [7]. \( M \) denotes the number of clusters, and \( Q \) the modularity; computing times are in seconds. Note that slight discrepancies may arise in the values of \( M \) and \( Q \); they are due to the fact that optimal bipartitions are not necessarily unique. It appears that the computing time is reduced by a factor of 2 to over 265.

<table>
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<tr>
<th>Network</th>
<th>( n )</th>
<th>( m )</th>
<th>( M )</th>
<th>( Q )</th>
<th>time</th>
<th>( M )</th>
<th>( Q )</th>
<th>time</th>
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</table>

TAB. 1: Results obtained with the hierarchical divisive heuristic using respectively the original formulation and the compact reformulation.

### References


