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# Reduced RLT constraints for polynomial programming

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ABSTRACT. An extension of the reduced Reformulation-Linearization Technique constraints from quadratic to general polynomial programming problems with linear equality constraints is presented and a strategy to improve the associated convex relaxation is proposed.

## 1. Introduction

Reduced RLT constraints (rRLT) are a special class of Reformulation-Linearization Technique (RLT) constraints, that apply to nonconvex (both continuous and mixed-integer) quadratic programming problems subject to linear equality constraints [2, 4, 3]. rRLT are obtained by replacing some of the quadratic terms with suitable linear constraints. These turn out to be a subset of the RLT constraints for quadratic programming [7].

We present an extension of the rRLT theory to the case of general polynomial programs. Then, we show a strategy to choose the basis of a matrix involved in the rRLT constraints generation so as to tighten the bound of the associated convex relaxation. This allows to improve the performance of a spatial Branch-and-Bound algorithm applied to nonconvex NLP and MINLP problems where such convex relaxation is computed at each node.

## 2. Extending rRLT to polynomial programs

Let  $n$  be the number of variables,  $q$  the degree of the polynomials in the targeted problem and  $\mathcal{N} = \{1, \dots, n\}$ ,  $Q = \{2, \dots, q\}$ . For each monomial  $x_{j_1} \cdots x_{j_p}$ ,  $p \in Q$ , appearing in the problem, we define a finite sequence  $J = (j_1, \dots, j_p)$  and consider defining constraints of the following form:

$$w_J = \prod_{\ell \leq |J|} x_{j_\ell} \quad (2.1)$$

(for  $|J| = 1$ , i.e.  $J = (j)$ , we also define  $w_J = x_j$ ). For all  $p \in Q$ ,  $J \in \mathcal{N}^p$  and any permutation  $\pi$  in the symmetric group  $S_p$  we have that  $w_J = w_{\pi J}$  by commutativity. We therefore define an equivalence relation  $\sim$  on  $\mathcal{N}^p$  stating that for  $J, K \in \mathcal{N}^p$ ,  $J \sim K$  only if  $\exists \pi \in S_p$  such that  $J = \pi K$ . We then consider the index tuple set  $\bar{\mathcal{N}}^p = \mathcal{N}^p / \sim$  to quantify over when indexing variables  $w_J$ .

We multiply the original linear constraints  $Ax = b$  by all monomials  $\prod_{\ell \leq p-1} x_{j_\ell}$  and replace them by the corresponding added variables  $w_{(J',j)}$ , where  $J' \in \bar{\mathcal{N}}^{p-1}$ . This yields the following rRLTS:

$$\forall p \in Q, J' \in \bar{\mathcal{N}}^{p-1} \quad A \mathbf{w}_{J'} = b w_{J'}, \quad (2.2)$$

where  $\mathbf{w}_{J'} = (w_{(J',1)}, \dots, w_{(J',n)})$ . We then consider the companion system:

$$\forall p \in Q, J' \in \bar{\mathcal{N}}^{p-1} \quad A \mathbf{z}_{J'} = 0. \quad (2.3)$$

Since (2.3) is a linear homogeneous system, there is a matrix  $M$  such that the companion system is equivalent to  $Mz = 0$ , the columns of which are indexed by sequences in  $\bar{\mathcal{N}}^p$ . We let  $B \subseteq \bar{\mathcal{N}}^p$

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and  $N \subseteq \bar{N}^p$  be index sets for basic and nonbasic columns of  $M$ . We define the following sets:

$$\begin{aligned} C &= \{(x, w) \mid Ax = b \wedge \forall p \in Q, J \in \bar{N}^p (w_J = \prod_{\ell \leq |J|} x_{j_\ell})\} \\ R_N &= \{(x, w) \mid Ax = b \wedge \forall p \in Q, J' \in \bar{N}^{p-1} (A \mathbf{w}_{J'} = bw_{J'}) \wedge \\ &\quad \forall J \in N (w_J = \prod_{\ell \leq |J|} x_{j_\ell})\}. \end{aligned}$$

**Theorem 1.** *For each partition  $B, N$  into basic and nonbasic column indices for the companion system  $Mz = 0$ , we have  $C = R_N$ .*

### 3. Tightening the convex relaxation

Replacing  $C$  with  $R_N$  for some nonbasis  $N$  effectively replaces some monomial terms with linear constraints, and therefore contributes to simplify the problem. A convex relaxation for the reformulated problem is readily obtained by applying monomial convexification methods in the literature [5, 6, 1]. We observe that for any given linear system there is in general more than one way to partition the variables in basics and nonbasics. Hence the set  $B$  can be chosen in such a way as to decrease the discrepancy between the feasible region and its convex relaxation. Given  $f : X \subseteq \mathbb{R}^n \rightarrow \mathbb{R}$  and the sets  $S = \{(x, w) \mid w = f(x)\}$  and  $\bar{S} = \{(x, w) \mid \underline{f}(x) \leq w \leq \bar{f}(x)\}$ , where  $\underline{f}(x)$ ,  $\bar{f}(x)$  are respectively a convex lower and a concave upper bounding function for  $f$  (and hence  $\bar{S}$  is a convex relaxation of  $S$ ), the *convexity gap* between  $S$  and  $\bar{S}$  can be defined as the volume  $V(S)$  of the set  $\bar{S}$ . Explicit expressions of  $V(S)$  can be derived for a quadratic term  $x_i^2$ , for a bilinear term  $x_i x_j$  using the Cayley-Menger formula in 3 dimensions, and for a general monomial, exploiting associativity recursively to rewrite it as product of lower degree monomials and using the preceding results.

Let  $B, N$  be the basic/nonbasic sets of column indices of the companion system, which we can write as  $M_B z_B + M_N z_N = 0$ . The elements of  $B, N$  are sequences  $J \in \mathcal{M}$ . For  $S \subseteq \mathcal{M}$  and  $p \in Q$  we define  $V^{S,p} = \sum_{\substack{J \in S \\ |J|=p}} V_J$  and  $V^S = \sum_{p \in Q} V^{S,p}$ . If, for all  $p \in Q$ ,  $V^{N,p} < V^{B,p}$  then the

total convexity gap of  $R_N$  is smaller than that of  $C$ . Thus, we aim to find  $N$  such that  $V^{N,p}$  is minimized, or equivalently, to find  $B$  such that  $V^{B,p}$  is maximized for all  $p \in Q$ . This yields the multi-objective problem:

$$\left. \begin{aligned} \forall p \in Q \quad \max V^{B,p} \\ M_B \text{ is a basis of (2.3)} \end{aligned} \right\} \quad (3.1)$$

It can be shown that (3.1) is equivalent to a single-objective problem: any solution  $B$  of (3.1) maximizing  $V^B$  also maximizes  $V^{B,p}$  for all  $p \in Q$ . In this way, we have derived a technique to choose a good basis for the companion system so as to improve the chances of tightening the lower bound of the convex relaxation associated to rRLT.

Preliminary computational experiments carried out on a set of randomly generated instances of the convex Quadratic Knapsack Problem (cQKP) show that the proposed strategy is promising in improving performances of a spatial Branch-and-Bound algorithm.

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