Rotorcraft Trajectory Tracking by Non Linear Inverse Control

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Abstract: The purpose of this communication is to investigate the usefulness of the non linear inverse control approach to solve the trajectory tracking problem for a four rotor aircraft. After introducing simplifying assumptions, the flight dynamics equations for the four rotor aircraft are considered. A trajectory tracking control structure based on a two layer non linear inverse approach is then proposed.A supervision level is introduced to take into account the actuator limitations.

Keywords: Rotorcraft flight mechanics, nonlinear inverse control, trajectory tracking.

1.INTRODUCTION

In the last years a large interest has risen for the four rotor concept since it appears to present simultaneously hovering, orientation and trajectory tracking capabilities of interest in many practical applications [1].

The flight mechanics of rotorcraft are highly non linear and different control approaches (integral LQR techniques, integral sliding mode control [2], reinforcement learning [3]) have been considered with little success to achieve not only autonomous hovering and orientation, but also trajectory tracking In this paper, after introducing some simplifying assumptions, the flight dynamics equations for a four rotor aircraft with fixed pitch blades are considered.

The purpose of this study is to investigate the usefulness of the non linear inverse control approach to solve the trajectory tracking problem for this class of rotorcraft. This approach has been already considered in the case of aircraft trajectory tracking by different authors [4,5,6].

It appears that the flight dynamics of the considered rotorcraft present a two level input affine structure which is made apparent when a new set of equivalent inputs is defined. This allows to introduce a non linear inverse control approach with two time scales, one devoted to attitude control and one devoted to orientation and trajectory tracking.

Rotor dynamics are discussed in annex 1 and since the control approach does not take into account explicitly the limitations of the actuators, a supervision layer is also proposed.

2. ROTORCRAFT FLIGHT DYNAMICS

The considered system is shown in figure 1 where rotors one and three are clockwise while rotors two and four are counter clockwise. Annex 1 describes the rotor dynamics.

The main simplifying assumptions adopted with respect to flight dynamics in this study are a rigid cross structure, no wind, negligible aerodynamic contributions resulting from translational speed, no ground effect as well as negligible air density effects and very small rotor response times. It is then possible to write simplified rotorcraft flight equations [7].

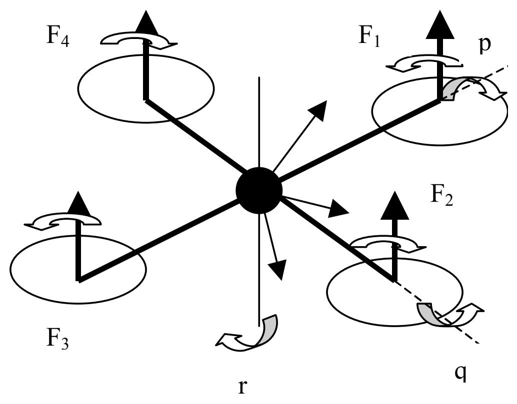

Figure 1- Four rotor aircraft

The moment equations can be written as:

$$\left.\begin{aligned} \dot{p} &= (a/I_{xx})\,(F_4 - F_2) + k_2\, q\, r \\ \dot{q} &= (a/I_{yy})\,(F_1 - F_3) + k_4\, p\, r \\ \dot{r} &= (k/I_{zz})\,(F_2 - F_1 + F_4 - F_3) \end{aligned}\right\} \quad (1)$$

where p, q, r are the components of the body angular velocity, with $k_2 (I_{zz} - I_{yy})/I_{xx}$ and $k_4 = (I_{xx} - I_{zz})/I_{yy}$, I_{xx}, I_{yy} and I_{zz} being the moments of inertia in body-axis and m the total mass of the rotorcraft.
The Euler equations are given by:

$$\left.\begin{aligned} \dot{\phi} &= p + \tan(\theta)\sin(\phi)\, q + \tan(\theta)\cos(\phi)\, r \\ \dot{\theta} &= \cos(\phi)\, q - \sin(\phi)\, r \\ \dot{\psi} &= ((\sin(\phi)/\cos(\theta))\, q + (\cos(\phi)/\cos(\theta))\, r \end{aligned}\right\} \quad (2)$$

where θ, ϕ, and ψ are respectively the pitch, bank and heading angles.

The acceleration equations written directly in the local Earth reference system are such as:

$$\left.\begin{aligned} \ddot{x} &= (1/m)(\cos(\psi)\sin(\theta)\cos(\phi) + \sin(\psi)\sin(\phi))\, F \\ \ddot{y} &= (1/m)(\sin(\psi)\sin(\theta)\cos(\phi) - \cos(\psi)\sin(\phi))\, F \\ \ddot{z} &= -g + (1/M)\cos(\theta)\cos(\phi)\, F \end{aligned}\right\} \quad (3)$$

where x, y and z are the centre of gravity coordinates and where :

$$F = F_1 + F_2 + F_3 + F_4 \quad (4)$$

and with the constraints:

$$0 \le F_{i_i} \le F_{\max} \quad i \in \{1,2,3,4\} \quad (5)$$

3. NLI FLIGHT CONTROL APPROACH

Here we are interested in controlling the four rotor aircraft so that its centre of gravity follows a given path with a given heading ψ while attitude angles θ and ϕ remain small. Many potential applications require not only the centre of gravity of the device to follow a given trajectory but also the aircraft to present a given orientation.

3.1 Attitude control

From equations (1) it appears that the effectiveness of the rotor actuators is much larger with respect to the roll and pitch axis than with respect to the yaw axis. Then we consider that attitude control is involved with controlling the θ and ϕ angles. In equations (1) the effect of rotor forces appears as differences so, we define new attitude inputs u_1 and u_2 as:

$$u_1 = F_1 - F_3 \quad u_2 = F_2 - F_4 \quad (6.1)$$

In the heading and position dynamics, the effects of rotor forces and moments appear as sums, so we define new guidance inputs v_1 and v_2 as:

$$v_1 = F_1 + F_3 \quad v_2 = F_2 + F_4 \quad (6.2)$$

It is supposed that u_1 and u_2 can be made to vary significantly while v_1 and v_2 can remain constant.
Then the attitude dynamics can be rewritten under the affine input form:

$$\underline{\dot{X}} = f(\underline{X}, \underline{V}) + g(\underline{X})\,\underline{U} \quad (7.1)$$

$$\underline{Y}' = (\theta, \phi) \quad (7.2)$$

with

$$\underline{X}' = (p, q, \theta, \phi)\,,\ \underline{U}' = (u_1, u_2) \text{ and } \underline{V}' = (v_1, v_2) \quad (8)$$

Then, considering the non linear inverse control theory, it appears that the attitude angles present relative degrees equal to one and that there is no internal dynamics while the output equations can be rewritten as:

$$\underline{\ddot{Y}} = M(\underline{Y})\,\underline{U} + N(\underline{X})\,\underline{V} + P(\underline{X}) \quad (9)$$

with

$$M(\underline{Y}) = \begin{bmatrix} a\, tg\theta \sin\phi / I_{yy} & -a/I_{xx} \\ a \cos\phi / I_{yy} & 0 \end{bmatrix} \quad (10)$$

$$N(\underline{X}) = \begin{bmatrix} k \sin\phi / I_{zz} & -k \sin\phi / I \\ -k\, tg\theta \cos\phi / I_{zz} & k\, tg\theta \cos\phi / I_{zz} \end{bmatrix} \quad (11)$$

$$P(\underline{X}) = [P_1, P_2]'$$

where:

$$P_1 = k_4 \cos\phi\; p\, r \quad (12.1)$$

$$-(q \sin\phi + r\cos\phi)\,(p + tg\theta\,(q \sin\phi + r\, tg\theta \cos\phi))$$

$$P_2 = k_2 q\, r + k_4\ p\ r\ tg\theta \sin\phi + q\ d(tg\theta \sin\phi)/dt + r\ d(tg\theta \cos\phi)/dt \quad (2.2)$$

It appears that while $\phi \neq \pm\pi/2$, the attitude dynamics given by (9) are invertible. Then it appears feasible to consider as control objective to get second order linear attitude dynamics towards reference values:

$$\begin{bmatrix} \ddot{\theta}_d \\ \ddot{\phi}_d \end{bmatrix} = \underline{\ddot{Y}}_d = \begin{bmatrix} -2\zeta_\theta\ \omega_\theta\ \dot{\theta} - \omega_\theta^{\,2}(\theta - \theta_c) \\ -2\zeta_\phi\ \omega_\phi\ \dot{\phi} - \omega_\phi^{\,2}(\phi - \phi_c) \end{bmatrix} \quad (13)$$

where $\zeta_\theta, \zeta_\phi, \omega_\theta, \omega_\phi$ are respectively damping and natural frequency parameters while θ_c and ϕ_c are reference values for the attitude angles.
Then the corresponding non linear inverse attitude control law is given by:

$$\underline{U} = -M(\underline{X})^{-1}(N(\underline{X})\ \underline{V} + P(\underline{X}) - \underline{\ddot{Y}}_d) \quad (14)$$

3.2 Design of a guidance control law

Considering that the attitude dynamics are stable and faster than the guidance dynamics, the guidance equations can be approximated by the control affine form:

$$\begin{bmatrix} \ddot{\psi} \\ \ddot{x} \\ \ddot{y} \\ \ddot{z} \end{bmatrix} = \begin{bmatrix} \dfrac{k \cos\phi_c}{I_{zz}\cos\theta_c}(v_2 - v_1) \\ (1/m)(\cos(\psi)\sin(\theta_c)\cos(\phi_c) + \sin(\psi)\sin(\phi_c))\,(v_1 + v_2) \\ (1/m)(\sin(\psi)\sin(\theta_c)\cos(\phi_c) - \cos(\psi)\sin(\phi_c))\,(v_1 + v_2) \\ -g + (1/m)\cos(\theta_c)\cos(\phi_c)\,(v_1 + v_2) \end{bmatrix} \quad (15)$$

Here also, the outputs of the guidance dynamics present relative degrees equal to 1 while the internal dynamics, which are concerned with the attitude angles , are supposed already stabilized. Then, supposing that second order linear dynamics are of interest for the guidance variables, we can define desired accelerations by:

$$\begin{bmatrix} \ddot{\psi}_c \\ \ddot{x}_c \\ \ddot{y}_c \\ \ddot{z}_c \end{bmatrix} = \begin{bmatrix} -2\,\zeta_\psi\ \omega_\psi\ \dot{\psi} - \omega_\psi^{\,2}(\psi - \psi_c) \\ -2\,\zeta_x\ \omega_x\ \dot{x} - \omega_x^{\,2}(x - x_c) \\ -2\,\zeta_y\ \omega_y\ \dot{y} - \omega_y^{\,2}(y - y_c) \\ -2\,\zeta_z\ \omega_z\ \dot{z} - \omega_z^{\,2}(z - z_c) \end{bmatrix} \quad (16)$$

where $\zeta_\psi, \zeta_x, \zeta_y, \zeta_z, \omega_\psi, \omega_x, \omega_y, \omega_z$ are respectively damping and natural frequency parameters while ψ_c, x_c, y_c and z_c *ar*e reference values for the attitude angles. Of course, many other schemes can be proposed to define desired accelerations at the guidance level.
Once desired accelerations are made available, the inversion of the guidance dynamics brings nominal the solution:

$$v_1 = \frac{1}{2}\left(m\sqrt{\ddot{x}_c^{\,2} + \ddot{y}_c^{\,2} + (\ddot{z}_c + g)^2} - \frac{I_{zz}\cos\theta_c}{k\,\cos\phi_c}\ddot{\psi}_c\right) \quad (17.1)$$

$$v_2 = \frac{1}{2}\left(m\sqrt{\ddot{x}_c^{\,2} + \ddot{y}_c^{\,2} + (\ddot{z}_c + g)^2} + \frac{I_{zz}\cos\theta_c}{k\,\cos\phi_c}\ddot{\psi}_c\right) \quad (17.2)$$

with

$$\theta_c = arctg((\ddot{x}_c \cos\psi + \ddot{y}_c \sin\psi)/(\ddot{z}_c + g)) \quad (18.1)$$

and

$$\phi_c = \arcsin\left(\frac{\ddot{x}_c \sin\psi - \ddot{y}_d \cos\psi}{\sqrt{\ddot{x}_d^{\,2} + \ddot{y}_d^{\,2} + (\ddot{z}_d + g)^2}}\right. \quad (18.2)$$

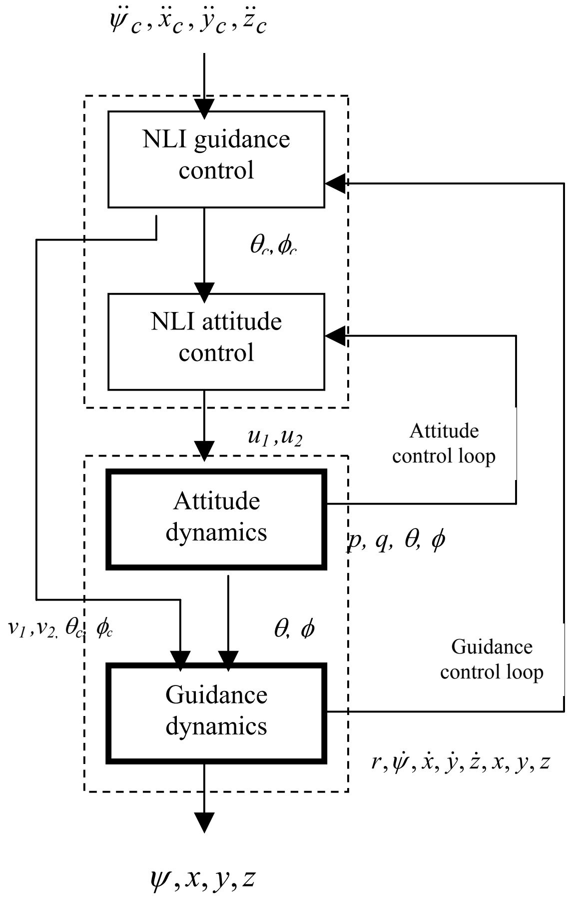

Figure 2-Proposed control structure

Then, returning to the expression of the attitude control law , it happens that the centre of gravity acceleration terms compensate each others and the law becomes:

$$\underline{U} = -M(\underline{X})^{-1}(\underline{N}(\underline{X}) + P(\underline{X}) - \underline{\ddot{Y}}_d) \quad (19)$$

with

$$\underline{N}(\underline{X}) = \begin{bmatrix} N_1 \\ N_2 \end{bmatrix} = \begin{bmatrix} -\cos\theta_c \dfrac{\sin\phi}{\cos\phi_c} \\ \sin\theta \dfrac{\cos\theta_c}{\cos\theta}\dfrac{\cos\phi}{\cos\phi_c} \end{bmatrix} \ddot{\psi}_c \quad (20)$$

The whole proposed control structure is given in the above figure 2.

4. FLIGHT CONTROL SUPERVISION

Since the above control approach does not consider explicitly the input level constraints, we introduce here a supervision layer whose function is to avoid the generation of unfeasible reference values for the inputs by modifying, as less as possible, the current control objectives. According to (5), (6) and (7), the control signals should be such as:

$$-F_{\max} \le u_i \le F_{\max} \quad i = 1,2 \qquad (21.1)$$

and

$$0 \le v_i \le 2\,F_{\max} \quad i = 1,2 \qquad (21.2)$$

Conditions (21.1) implies for the desired attitude accelerations to satisfy the following conditions:

$$\ddot{\phi}_{\min} \le \ddot{\phi}_c \le \ddot{\phi}_{\max} \qquad (22.1)$$

with

$$\ddot{\phi}_{\min} = N_2 + P_2 - \frac{a\cos\phi}{I_{yy}} F_{\max} \qquad (22.2)$$

and

$$\ddot{\phi}_{\max} = N_2 + P_2 + \frac{a\cos\phi}{I_{yy}} F_{\max} \qquad (22.3)$$

and condition:

$$\ddot{\theta}_{\min} \le \ddot{\theta}_c - tg\theta\,\ddot{\phi}_c \le \ddot{\theta}_{\max} \qquad (23.1)$$

with

$$\ddot{\theta}_{\min} = -a\,F_{\max} / I_{xx} + (N_1 + P_1) - tg\theta\,(N_2 + P_2) \qquad (23.2)$$

and

$$\ddot{\theta}_{\max} = a\,F_{\max} / I_{xx} + (N_1 + P_1) - tg\theta\,(N_2 + P_2) \qquad (23.3)$$

Then, reference values for instant attitude angles accelerations can be obtained from the solution of the following linear –quadratic optimization problem:

$$\min_{\alpha, \beta} \; (\ddot{\theta}_c - \alpha)^2 + (\ddot{\phi}_c - \beta)^2 \qquad (24.1)$$

with

$$\ddot{\phi}_{\min} \le \beta \le \ddot{\phi}_{\max} \qquad (24.2)$$

$$\ddot{\theta}_{\min} \le \alpha - tg\theta\,\beta \le \ddot{\theta}_{\max} \qquad (24.3)$$

Observe that the solution of this problem is equal to $(\ddot{\theta}_c, \ddot{\phi}_c)$ if it is feasible with respect to constraints (24.2) and (24.3), otherwise the solution will be on the border of the convex feasible set.

Then if α^* and β^* are solution of this problem, u_1 and u_2 are given by:

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = -M(\underline{X})^{-1}(N(\underline{X})\,\underline{V} + P(\underline{X}) - \begin{bmatrix} \alpha^* \\ \beta^* \end{bmatrix}) \qquad (25)$$

In the case of v_1 and v_2 (relations (21.2)) and considering the expressions of θ_c and ϕ_c the above approach leads to the consideration of an intricate non convex optimization problem. A different approach is proposed here. Let λ be such as:

$$\ddot{x}_r = \lambda\,\ddot{x}_c,\; \ddot{y}_r = \lambda\,\ddot{y}_c,\; \ddot{z}_r + g = \lambda(\ddot{z}_c + g) \qquad (26)$$

then according to (18.1) and (18.2):

$$\theta_r = \theta_c \quad \text{and} \quad \phi_r = \phi_c \qquad (27)$$

Feasible reference values for $\ddot{x}_r, \ddot{y}_r, \ddot{z}_r$ and $\ddot{\psi}_r$ can be obtained from the solution of the following linear – quadratic optimization problem:

$$\min_{\lambda,\mu} (\lambda - 1)^2 + \eta^2(\mu - \ddot{\psi}_c)^2 \qquad (28.1)$$

with

$$0 \le (m\sqrt{\ddot{x}_c^{\,2} + \ddot{y}_c^{\,2} + (\ddot{z}_c + g)^2})\,\lambda - (\frac{I_{zz}\cos\theta_c}{k\,\cos\phi_c})\,\mu \le 4\,F_{\max} \qquad (28.2)$$

$$0 \le (m\sqrt{\ddot{x}_c^{\,2} + \ddot{y}_c^{\,2} + (\ddot{z}_c + g)^2})\,\lambda + (\frac{I_{zz}\cos\theta_c}{k\,\cos\phi_c})\,\mu \le 4\,F_{\max} \qquad (28.3)$$

where η is here a time constant. Let λ^* and μ^* be the solution of the above problem, then the control inputs can be taken as:

$$v_1 = \frac{1}{2}(m\,\lambda^* \sqrt{\ddot{x}_c^{\,2} + \ddot{y}_c^{\,2} + (\ddot{z}_c + g)^2} - \frac{I_{zz}\cos\theta_c}{k\,\cos\phi_c}\,\mu^*) \qquad (29.1)$$

$$v_2 = \frac{1}{2}(m\,\lambda^* \sqrt{\ddot{x}_c^{\,2} + \ddot{y}_c^{\,2} + (\ddot{z}_c + g)^2} + \frac{I_{zz}\cos\theta_c}{k\,\cos\phi_c}\,\mu^*) \qquad (29.2)$$

Then:

$$F_1 = (u_1 + v_1)/2 \quad F_2 = (u_2 + v_2)/2 \qquad (30.1)$$

$$F_3 = (v_1 - u_1)/2 \quad F_4 = (v_2 - u_2)/2 \qquad (30.1)$$

5. CASE STUDIES

Here we consider two cases: one where the objective is to hover at an initial position of coordinates x_0, y_0, z_0 while acquiring a new orientation ψ_1, and one where the rotorcraft is tracking the helicoïdal trajectory of equations:

$$\left.\begin{array}{c} x_c(t) = \rho\cos\nu t \\ y_c(t) = \rho\sin\nu t \\ z_c = \delta + \gamma t \\ \psi_c(t) = \nu t + \pi/2 \end{array}\right\} \qquad (31)$$

where ρ is a constant radius and γ is a constant path angle.

5.1 Heading control at hover

In this case we get the guidance control laws:

$$v_1 = \frac{1}{2}(m\,g - \frac{I_{zz}}{k}\ddot{\psi}_c) \quad v_2 = \frac{1}{2}(m\,g + \frac{I_{zz}}{k}\ddot{\psi}_c) \qquad (32)$$

with the following reference values for the attitude angles:

$$\theta_c = 0 \quad \text{and} \quad \phi_c = 0 \qquad (33)$$

Here the heading acceleration is given by:

$$\ddot{\psi}_c = -2\,\zeta_\psi\,\omega_\psi r - \omega_\psi{}^2(\psi - \psi_1) \qquad (34)$$

Starting from an horizontal attitude ($\theta(0)$=0, $\phi(0)$=0), attitude inputs u_1 and u_2 given by relation (14) remain equal to zero. Then, figures 3 and 4 display some simulation results:

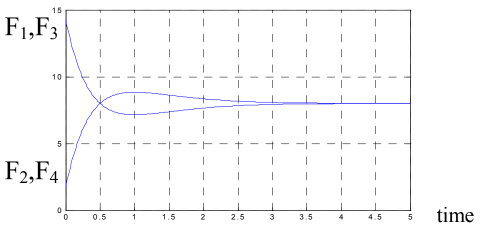

Figure 3-Hover control inputs

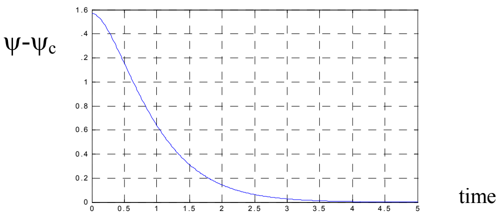

Figure 4-Heading response during hover

5.2 Trajectory tracking case

In this case we get the guidance control laws:

$$v_1 = v_2 = \frac{1}{2}(m\sqrt{\rho^2 v^2 + g^2}) \qquad (38)$$

Here the permanent reference values for the attitude angles are such as:

$$\theta_c = 0 \qquad (39.1)$$

and

$$\sin\phi_c = -\frac{\rho\,v^2}{\sqrt{\rho^2 v^4 + g^2}} \qquad (39.2)$$

and the desired guidance and orientation accelerations are given by:

$$\left.\begin{aligned} \ddot{x}_c &= -\rho\,v^2\cos(v\,t) \\ \ddot{y}_c &= -\rho\,v^2\sin(v\,t) \\ \ddot{z}_c &= 0, \quad \ddot{\psi}_c = 0 \end{aligned}\right\} \qquad (40)$$

Attitude inputs are given by relation (14) where now:

$$M^{-1} = \begin{bmatrix} 0 & I_{yy}/(a\cos\phi) \\ -1/(a\,I_{xx}) & tg\theta\,tg\phi/(a\,I_{xx}) \end{bmatrix} \qquad (41)$$

and

$$\underline{N}(\underline{X})' = \begin{bmatrix} 0 & 0 \end{bmatrix} \qquad (41)$$

In figures 5 to 7 simulation results are displayed where at initial time the rotorcraft is hovering:

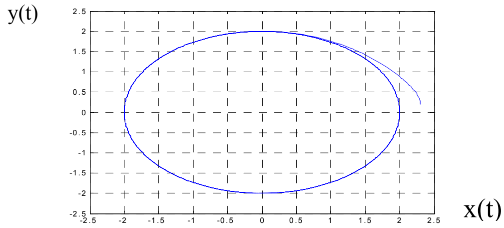

Figure 5- Evolution of rotorcraft horizontal track

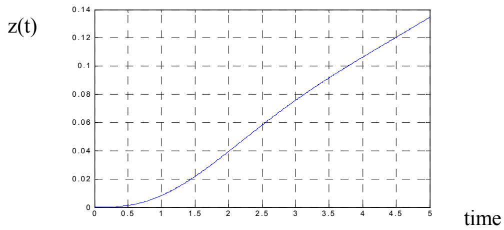

Figure 6- Evolution of rotorcraft altitude

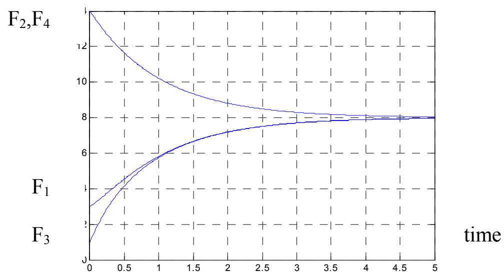

Figure 7- Rotorcraft trajectory tracking inputs

6. CONCLUSIONS

In this communication the theoretical applicability of the non linear inverse control technique to rotorcraft trajectory tracking has been investigated. It appears that this approach leads to the design of a two level control structure based on analytical laws.

Considering the structure of the rotorcraft flight dynamics, other promising non linear control techniques are differential flat control [8] and back stepping control [9].

When considering the complexity of these non linear control laws involving a relatively small number of inputs, neural networks components could be of interest for their effective implementation.

However, the robustness of these control laws with respect to the different aerodynamic effects which have been taken as negligible should be investigated. Since only very intricate theories are available to approach this problem, real flight tests appear, at this stage, to be unavoidable.

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Annex 1.

The rotor engine dynamics are characterized by the relation between the input voltage V_a and the angular rate ω. A possible model of rotor dynamics is given by:

$$\dot{\omega}(t) = -\frac{1}{\tau}\omega(t) - K_Q\omega(t)^2 + (K_{V_a}/\tau)\,V_a(t) \qquad (A.1)$$

with $\omega(0) = \omega_0$, where τ , K_Q and K_{Va} are given positive parameters and where the voltage input is such as:

$$0 \le V_a \le V_{\max} \qquad (A.2)$$

with a negligible time response for the voltage generator.

The step response (V_a =constant) of the rotor is solution of the scalar *Riccati* equation:

$$\dot{\omega}(t) = -\frac{1}{\tau}\omega(t) - K_Q\omega(t)^2 + (K_{V_a}/\tau)\,V_a \qquad (A.3)$$

with $\omega(0) = \omega_0$.

A particular solution ω_1 of the associated differential equation is such as:

$$\omega_1 = \frac{1}{2\,\tau\,K_Q}\left(\sqrt{1 + 4K_{V_a}K_Q\tau\,V_a} - 1\right) \qquad (A.4)$$

In the general case, the solution of (A.3) can be written as

$$\omega(t) = \omega_1 + \frac{1}{\dfrac{1}{\omega(0)-\omega_1} + K_Q\tau'(1-e^{-t/\tau'})}\,e^{-t/\tau'} \qquad (A.5)$$

with

$$\tau' = \tau / \sqrt{1 + 4K_V K_Q \tau\, V_a} \qquad (A.6)$$

and

$$\lim_{t\to+\infty} \omega(t) = \omega_1 \qquad (A.7)$$

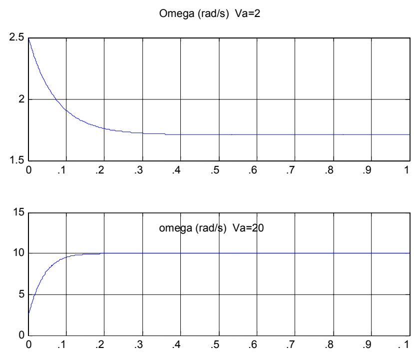

Figure 8- Two examples of rotor step response

It appears from figure 8 that the dynamics of the rotor may be close to those of a first order linear system with time constant τ', but as can be seen in (A.6), this value is a function of V_a.

If the desired dynamics for the output are such as:

$$\dot{\omega} = -\frac{1}{T}(\omega - \omega_c) \qquad (A.8)$$

where T is a very small time constant V_a can be chosen such as:

$$V_a(t) = \frac{1}{K_{V_a}}\left(\left(1-\frac{\tau}{T}\right)\omega(t) + \frac{\tau}{T}\omega_c + \tau\,K_Q\,\omega(t)^2\right) \qquad (A.9)$$

The rotor forces are then given by:

$$F_i = f\,\omega_i^2 \qquad i = 1\ to\ 4 \qquad (A.10)$$

while the rotor moments are given by:

$$M_i = k\ F_i \qquad i = 1\ to\ 4 \qquad \text{(A.11)}$$