Differential Flatness and Control of Nonlinear Systems

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Abstract: The purpose of this communication is to investigate the connection between output relative degrees and differential flatness property of non linear systems. Considering the relative degrees of the outputs of a non linear system, necessary and sufficient conditions for differential flatness are displayed. Then, it is shown that the application of the non linear inverse approach to an output observable and output differential flat system is identical with the differential flat control approach. The application of the non linear control approach to a four rotor aircraft is considered. A trajectory tracking system based on a two layers non linear inverse control structure is then proposed.

Key Words: Differential Flatness, Nonlinear Inverse Control, Trajectory Tracking

1 Introduction

In the last decade a large interest has risen for new non linear control approaches such as non linear inverse control [1,2,3], backstepping control [4] and differential flat control [5,6]. These control law design approaches present some strong similarities which have remained unclear until today. In this communication relations between the non linear control approach and the differential flat control approaches are tackled through the consideration of the relative degrees of the tracked outputs. Additional assumptions and definitions appear to be opportune in this study, such as effect independency of the inputs of a general non linear system, output observability and output controllability of a differential flat system.

The application considered in this study is about trajectory tracking by a four rotor aircraft. The flight mechanics of rotorcraft are highly non linear and different control approaches (integral LQR techniques, integral sliding mode control) have been considered with little success to achieve not only autonomous hovering and orientation, but also trajectory tracking

More recently nonlinear analytical control design techniques have been applied to rotorcraft trajectory tracking [7,8]. It appears that the flight dynamics of the rotorcraft present a two level differential flat structure which is made apparent when a new set of equivalent inputs is defined. This allows to introduce here a non linear inverse control approach with two time scales, one devoted to attitude, heading and altitude control and one devoted to horizontal trajectory tracking.

2 Differential Flat Output and Control

Consider a general non-linear dynamic continuous system given by:

$$\underline{\dot{X}} = f(\underline{X},\underline{U}) \qquad (1)$$

$$\underline{Y} = h(\underline{X}) \qquad (2)$$

where $\underline{X}\in R^n$, $\underline{U}\in R^m$, $\underline{Y}\in R^m$, f is a smooth vector field of $\underline{X}$ and $\underline{U}$ and h is a smooth vector field of $\underline{X}$.

It is supposed here that the considered inputs are independent which means that each input has an *independent effect* over the state dynamics:

$$rank\left[\partial f / \partial u_i, \cdots, \partial f / \partial u_m\right] = m \qquad (3)$$

so that $\underline{U}$ can be extracted from (1) and it is possible to write:

$$\underline{U} = g(\underline{X},\underline{\dot{X}}) \qquad (4)$$

where g is a smooth function.

2.1 Relative degrees of outputs for nonlinear systems

According to [1] the system (1)-(2) is said to have with respect to each independent output Y_i, a relative degree r_i if the output dynamics can be written as:

$$\begin{pmatrix} Y_1^{(r_1+1)} \\ \vdots \\ Y_m^{(r_m+1)} \end{pmatrix} = \begin{bmatrix} b_1(\underline{X},\underline{U}) \\ \vdots \\ b_m(\underline{X},\underline{U}) \end{bmatrix} \qquad (5)$$

with $$Y_j^{(s)} = a_{js}(\underline{X}) \quad s = 0,\cdots,r_j,\ j = 1,\cdots,m \qquad (6)$$

and $$\partial b_j(\underline{X},\underline{U}) / \partial \underline{U} \neq \underline{0} \quad j = 1,\cdots,m \qquad (7)$$

The output dynamics (5)-(6) can be rewritten globally as:

$$\underline{\tilde{\tilde{Y}}} = A(\underline{X}) \qquad (8)$$

and $$\underline{\tilde{Y}} = B(\underline{X},\underline{U}) \qquad (9)$$

where $$\underline{\tilde{\tilde{Y}}} = (Y_1 \; \cdots \; Y_1^{(r_1)} \; \cdots \; Y_m \; \cdots \; Y_m^{(r_m)})' \quad (10)$$

and $$\underline{\tilde{Y}} = (Y_1^{(r_1+1)}, \cdots, Y_m^{(r_m+1)})' \quad (11)$$

Here

$$\underline{A}(\underline{X}) = \begin{bmatrix} \underline{a}_1(\underline{X}) \\ \vdots \\ \underline{a}_m(\underline{X}) \end{bmatrix} \quad (12\text{-}1)$$

with $$\underline{a}_j(\underline{X}) = \begin{bmatrix} a_{j0}(\underline{X}) \\ \vdots \\ a_{j,r_j}(\underline{X}) \end{bmatrix} \quad j = 1, \cdots, m \quad (12\text{-}2)$$

The relative degrees obey (see [2]) to the condition:

$$\sum_{i=1}^{m} r_i \leq n - m \quad (13)$$

When the strict equality holds, vector $\underline{\tilde{\tilde{Y}}}$ can be adopted as a new state vector for system (1), otherwise internal dynamics must be considered. Then any control law based on output feedback will be unable to master these internal dynamics and if these internal dynamics are unstable, the control scheme will be inappropriate. When internal dynamics are stable or don't exist, it will be worth to consider an output feedback control law. From (9), while $B(\underline{X}, \underline{U})$ is invertible with respect to $\underline{U}$, an output feedback control law such as:

$$\underline{U}(\underline{X}) = B_u^{-1}(\underline{X})\, \underline{\tilde{Y}} \quad (14)$$

can be adopted.

2.2 Differential flat systems

Now suppose that $\underline{Z} \in R^m$ is a differential flat output for system (1), then from [3] the state and the input vectors can be written as:

$$\underline{X} = \eta(\underline{\tilde{\tilde{Z}}}) \quad (15\text{-}1)$$

$$\underline{U} = \xi(\underline{\tilde{\tilde{Z}}}, \underline{\tilde{Z}}) \quad (15\text{-}2)$$

with

$$\underline{\tilde{\tilde{Z}}} = (Z_1, \cdots, Z_1^{(s_1)}, \cdots, Z_m, \cdots, Z_m^{(s_m)})' \quad (16\text{-}1)$$

and

$$\underline{\tilde{Z}} = (Z_1^{s_1+1}, Z_2^{s_2+1} \cdots, Z_m^{s_m+1})' \quad (16\text{-}2)$$

where η (.) is a function of Z_j and its derivatives up to order s_j, and ξ (.) is a function of Z_j and its derivatives up to order s_j+1, for j = 1 to m where the s_j are integers.
It appears of interest to introduce here three new definitions:

The differential flat system is said *output observable* if :

$$rank\left(\left[\partial\eta / \partial\underline{\tilde{\tilde{Z}}}\right]\right) = n \quad (17\text{-}1)$$

The differential flat system is said *full flat differential* if:

$$\sum_{i=1}^{m} s_i = n - m \quad (17\text{-}2)$$

The differentiable flat system (1) is said *output controllable* if:

$$\det([\partial\xi / \partial\underline{\tilde{Z}}]) \neq 0 \quad (17\text{-}3)$$

In that case too, it is easy to derive a control law of order s_j+1 with respect to output j by considering an output dynamics such as:

$$\underline{\tilde{Z}} = C(\underline{\tilde{\tilde{Z}}}, \underline{V}) \quad (18\text{-}1)$$

where C is such that the dynamics of $\underline{Z}$ are stable and where $\underline{V} \in R^m$ is an independent input . Then:

$$\underline{U} = \xi(\underline{\tilde{\tilde{Z}}}, C(\underline{\tilde{\tilde{Z}}}, \underline{V})) \quad (18\text{-}2)$$

3 Necessary and Sufficient Condition for Output Differential Flatness

3.1 Flatness and internal dynamics

It appears from relations (8) and (9) that a sufficient condition for system (1) to be differentially flat output observable and output controllable with respect to $\underline{Y}$ given by (2) is that A is invertible with respect to $\underline{X}$ and that B is invertible with respect to $\underline{U}$.

A necessary and sufficient condition for the invertibility of A is:

$$\sum_{i=1}^{m} r_i = n - m \quad (19)$$

while (3) is a necessary condition for the invertibility of B with respect to $\underline{U}$. In that case it is possible to define functions η and ξ by:

$$\underline{X} = A^{-1}(\underline{\tilde{\tilde{Y}}}) = \eta(\underline{Y}, \underline{\dot{Y}}, \cdots, \underline{Y}^{(p)}) \quad (20)$$

and

$$\underline{U} = B_u^{-1}(A^{-1}(\underline{\tilde{\tilde{Y}}}))(\underline{\tilde{Y}}) = \xi(\underline{Y}, \underline{\dot{Y}}, \cdots, \underline{Y}^{(p+1)}) \quad (21)$$

Here:

$$s_j = r_j \quad j = 1 \text{ to } m \quad (22)$$

Then we have got here a practical way to check if a given output vector $\underline{Y}$ is a differential flat output: it should be such as the corresponding matrices A and B are respectively global invertible and invertible with respect to $\underline{U}$, while condition (19) should be satisfied.
Then, a sufficient condition for differential flatness of $\underline{Z}$ is that $\underline{\tilde{\tilde{Z}}}$ is a state vector for system (1), i.e. there are no internal dynamics in this case.

3.2 Relative degree of a flat output

Suppose now that system (1) is a differential flat output observable and output controllable system where $\underline{Z}$ are the flat outputs with relative degrees r_j, j=1 to m. Then for a full flat differential system , from (15-1) and (16-1) we can write:

$$\underline{\tilde{\tilde{Z}}} = \Lambda(\underline{X}) \quad (23)$$

where Λ is a mapping from R^n to R^n . Then, from (15-2) and (16-2) we can write:

$$\underline{\widetilde{Z}} = \Gamma(\underline{\widetilde{\widetilde{Z}}}, \underline{U}) \tag{24}$$

where Γ is a mapping from R^n to R^m . Then, taking into account (23):

$$\underline{\widetilde{Z}} = E(\underline{X}, \underline{U}) = \begin{bmatrix} e_1(\underline{X}, \underline{U}) \\ \vdots \\ e_m(\underline{X}, \underline{U}) \end{bmatrix} \tag{25}$$

where E is a mapping from R^{n+m} to R^m.

Then, comparing (23) and (24) with (8) and (9), all the relative degrees r_j are superior or equal to the corresponding s_j. Suppose now that for some $j \in \{1, \cdots, m\}$ we have $\partial e_j / \partial \underline{U} = 0$ then r_j is necessarily strictly superior to s_j and we should have:

$$\sum\nolimits_{i=1}^{m} r_i > \sum\nolimits_{i=1}^{m} s_i \tag{26}$$

Considering (13) and (17-2), this is impossible. Then for a full flat differential system we have necessarily:

$$r_i = s_i \quad i = 1, \cdots, m \tag{27}$$

3.3 Output feedback control for trajectory tracking

From the above considerations it appears that there is no difference between a differential flat control law and a non linear inverse control law when applied to an output observable and output controllable differential flat system with the same control objectives for the respective outputs. Then, coming back to relation (14), and supposing that the nonlinear system (1)-(2) is an output observable and output controllable differential flat system, a new control input $\underline{v} = [v_1, \ldots, v_m]'$ can be introduced in place of $\underline{\widetilde{Y}}$ such as:

$$v_j = Y_{dj}^{(r_j)} - \sum_{k=0}^{r_j - 1} c_{jk} \left(Y_j^{(k)} - Y_{dj}^{(k)} \right) \quad j=1 \text{ to } m \tag{28}$$

making the j^{th} output to follow linear dynamics of order $r_j + 1$ towards the target value Y_{dj} . This makes the dynamics of the tracking error given by:

$$e_j = Y_j - Y_{dj} \quad \text{j=1 to } m \tag{29}$$

be such as:

$$e_i^{(r_i)} + c_{ir_i-1} e_i^{(r_i-1)} + \cdots + c_{i1} e_i^{(1)} + c_{i0} e_i = 0 \tag{30}$$

where the coefficients c_{ik} can be chosen to make the output dynamics asymptotically stable and ensure the tracking of output Y_i towards the reference output Y_{di}. Observe that in the present case relation (20) holds and there are no internal dynamics. To cope with the saturation of the actuators, the choice of the coefficients c_{ik} should be the result of a trade-off between the characteristics of the transient dynamics of the different outputs and the extreme solicitations of the inputs.
An output non observable differential flat system, when controlled through output feedback will present non controlled internal dynamics. While an output non controllable flat system will be unable to make its outputs follow, through an output feedback control, decoupled linear dynamics of order s_j+1, j = 1 to m.

4 Differential Flatness of Rotorcraft Dynamics

The considered system is shown in figure 1 where rotors one and three are clockwise while rotors two and four are counter clockwise. The main simplifying assumptions adopted with respect to flight dynamics in this study are a rigid cross structure, no wind, negligible aerodynamic contributions resulting from translational speed, no ground effect as well as negligible air density effects and very small rotor response times. It is then possible to write simplified rotorcraft flight equations [7].
The rotor forces and moments for the rotorcraft displayed in figure 8 are given by:

$$F_i = f\ \omega_i^2 \qquad i \in \{1, 2, 3, 4\} \tag{31}$$

$$M_i = k\ F_i = k\ f\ \omega_i^2 \qquad i \in \{1, 2, 3, 4\} \tag{32}$$

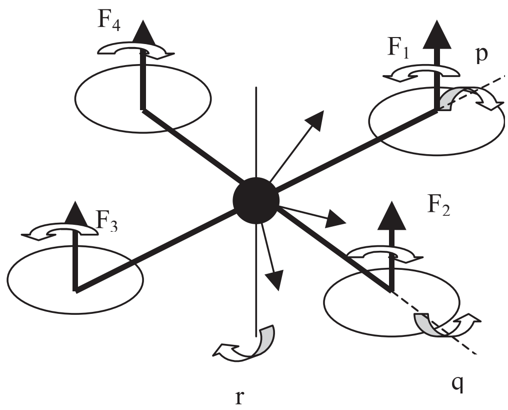

Fig. 1: Reference frame and forces of a four rotorcraft

where f and k are positive constants and ω_i is the rotational speed of rotor i. Since the inertia matrix of the rotorcraft can be considered diagonal with $I_{xx} = I_{yy}$, the roll, pitch and yaw moment equations may be written as:

$$\dot{p} = (l\ (F_4 - F_2) + k_2\ q\ r) / I_{xx} \tag{33-1}$$

$$\dot{q} = (l\ (F_1 - F_3) + k_4\, p\ r) / I_{yy} \tag{33-2}$$

$$\dot{r} = (k\ (F_2 - F_1 + F_4 - F_3)) / I_{zz} \tag{33-3}$$

Where p, q and r are the roll, pitch and yaw body angular rates. Here $k_2 = (I_{zz} - I_{yy})$ and $k_4 = (I_{xx} - I_{zz})$, where I_{xx}, I_{yy} and I_{zz} are the inertia moments in body-axis, and l is the length of the four arms of the rotorcraft.
ϕ, θ and ψ are respectively the bank, pitch and heading angles, then the Euler equations relating the derivatives of the attitude angles to the body angular rates, are given by:

$$\dot{\phi} = p + tg(\theta)(\sin\phi\ \ q + \cos\phi\ \ r) \tag{34-1}$$

$$\dot{\theta} = \cos\phi\ \ q - \sin\phi\ \ r \tag{34-2}$$

$$\dot{\psi} = (\sin\phi\, q + \cos\phi\, r) / \cos\theta \tag{34-3}$$

In this study it is assumed that there is no wind. The acceleration $\underline{a} = (a_x \quad a_y \quad a_z)'$ of the centre of gravity, taken directly in the local Earth reference frame, is such as:

$$a_x = (1/m)((\cos(\psi)\sin(\theta)\cos(\phi) + \sin(\psi)\sin(\phi))\ F) \quad (35\text{-}1)$$

$$a_y = (1/m)((\sin(\psi)\sin(\theta)\cos(\phi) - \cos(\psi)\sin(\phi))\ F) \quad (35\text{-}2)$$

$$a_z = g - (1/m)(\cos(\theta)\cos(\phi)\ F) \quad (35\text{-}3)$$

where *x, y* and *z* are the centre of gravity coordinates, m is the total mass of the rotorcraft and:

$$F = F_1 + F_2 + F_3 + F_4 \quad (36)$$

In equations (33-1) and (33-2), the effect of the rotor forces appears as differences so, we define new attitude inputs u_q and u_p as:

$$u_q = F_1 - F_3 \quad (37\text{-}1)$$

$$u_p = F_4 - F_2 \quad (37\text{-}2)$$

In the heading and position dynamics, the effects of rotor forces and moments appear as sums, so we define new guidance inputs u_ψ and u_z as:

$$u_\psi = (F_2 + F_4) - (F_1 + F_3) \quad (37\text{-}3)$$

$$u_z = F = F_1 + F_2 + F_3 + F_4 \quad (37\text{-}4)$$

Equations 33-1, 33-2 and 33-3 are rewritten:

$$\dot{p} = (l\ u_p + k_2\ q\ r)/I_{xx} \quad (38\text{-}1)$$

$$\dot{q} = (l\ u_q + k_4 p\ r)/I_{yy} \quad (38\text{-}2)$$

$$\dot{r} = k\ u_\psi / I_{zz} \quad (38\text{-}3)$$

Finally, the motion equations of the rotorcraft can be written in non-linear state form as:

$$\dot{\underline{x}} = f(\underline{x}, \underline{u}) \quad (39\text{-}1)$$

$$\text{where} \quad \underline{x} = (p, q, r, \phi, \theta, \psi, \dot{x}, \dot{y}, \dot{z}, x, y, z)' \quad (39\text{-}2)$$

$$\text{and} \quad \underline{u} = (u_p, u_q, u_\psi, u_z)' \quad (39\text{-}3)$$

It appears that controls u_q and u_r can be made to vary significantly with u_ψ and u_z remaining constant. Attitude angles ϕ and θ can be seen as virtual controls for the horizontal position of the rotorcraft. Here the attitude dynamics are considered to be the fast dynamics, they are at the heart of the control system. The heading and height dynamics are intermediate while the dynamics of the horizontal position coordinates are the slower. This can lead to a two-level closed-loop control structure. In this two level control structure, the final outputs are the coordinates of the center of gravity of the rotorcraft x, y, z and its heading ψ while the intermediate outputs are given by vector $\underline{Z} = (\phi, \theta, \psi, z)'$.

Then the Euler equations provides the expressions :

$$p = \dot{\phi} - \sin\theta\ \dot{\psi} \quad (40\text{-}1)$$

$$q = \cos\phi\ \dot{\theta} + \sin\phi\ \cos\theta\ \dot{\psi} \quad (40\text{-}2)$$

$$r = -\sin\phi\ \dot{\theta} + \cos\phi\ \cos\theta\ \dot{\psi} \quad (40\text{-}3)$$

while $\underline{u}$ can be expressed by inversion of the set of equations (38-1), (38-2), (38-3) and (35-3), or more specifically:

$$u_p = (-I_{xx}\dot{p} + k_2 qr)/l \quad (41\text{-}1)$$

$$u_q = (I_{yy}\dot{q} - k_4 pr)/l \quad (41\text{-}2)$$

$$u_\psi = -(I_{zz}\dot{r})/k \quad (41\text{-}3)$$

$$u_z = -((\ddot{z} - g)m)/(\cos\theta\cos\phi) \quad (41\text{-}4)$$

Then, it can be concluded that the attitude and heading dynamics as well as the vertical dynamics of the rotorcraft are differentially flat when considering the input-output relation between $\underline{u}$ and $\underline{Z}$. Here, the relative degrees of θ, ϕ, ψ and z are all equal to 2 while the dimension of the attitude, heading and altitude dynamics are of the 8th order, then relations (20) holds while it can be easily shown that these differential flat dynamics are output observable and output controllable.

When considering outputs x and y from entries θ and ϕ, where ψ and z play the role of parameters, it appears from equations (35-1) and (35-2) that these slow dynamics are also output observable and output controllable differential flat with relative degrees equal to two for a 4th order dynamics. This leads to propose the control structure displayed in figure 2 .

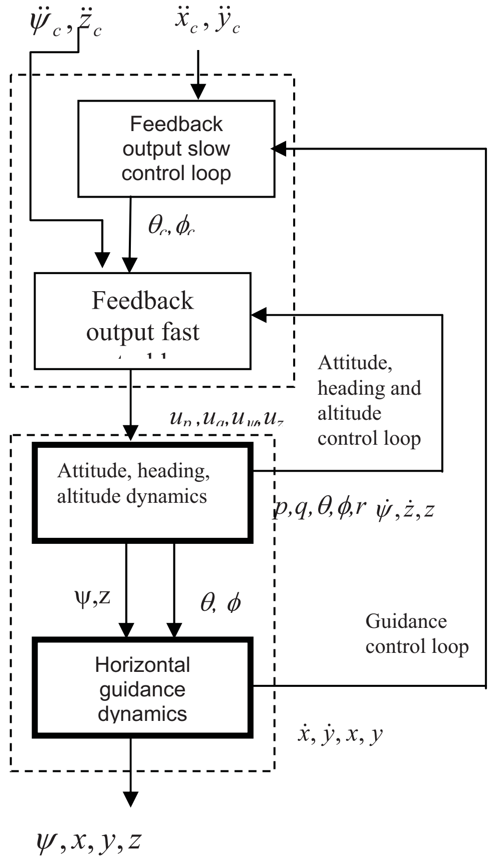

Fig. 2: Proposed control structure

5 Rotorcraft Trajectory Tracking

Here we are interested in controlling the four rotor aircraft of figure 3 so that its centre of gravity follows a given path with a given heading ψ while attitude angles θ and ϕ remain small. Many potential applications require not only the centre of gravity of the device to follow a given trajectory but also the aircraft to present a given orientation.

5.1 Fast dynamics control

We adopt for the flat outputs second order dynamics and their second derivative should be such as:

$$\ddot{\phi}_d = -2\,\zeta_\phi\,\omega_\phi\,\dot{\phi} - \omega_\phi^{\,2}(\phi - \phi_{cc}) \tag{42-1}$$

$$\ddot{\theta}_d = -2\,\zeta_\theta\,\omega_\theta\,\dot{\theta} - \omega_\theta^{\,2}(\theta - \theta_{cc}) \tag{42-2}$$

$$\ddot{\psi}_d = -2\,\zeta_\psi\,\omega_\psi\,\dot{\psi} - \omega_\psi^{\,2}(\psi - \psi_c) \tag{42-3}$$

$$\ddot{z}_d = -2\,\zeta_z\,\omega_z\,\dot{z} - \omega_z^{\,2}(z - z_c) \tag{42-4}$$

Fig 3: The considered rotorcraft

The expressions of the control inputs in relations (41-1) to (41-4) are fed by p, q, r given by relations (40-1) to (40-3) and by $\dot{p}, \dot{q}, \dot{r}$ given by:

$$\dot{p} = \ddot{\phi}_d - \cos\theta\,\dot{\theta}\,\dot{\psi} - \sin\theta\,\ddot{\psi}_d \tag{43-1}$$

$$\begin{aligned}\dot{q} &= \cos\phi\,\ddot{\theta}_d + \sin\phi\,\cos\theta\,\ddot{\psi}_d \\ &\quad -\sin\phi\,(1+\sin\theta)\,\dot{\theta}\,\dot{\psi} + \cos\phi\,\cos\theta\,\dot{\phi}\,\dot{\psi}\end{aligned} \tag{43-2}$$

$$\begin{aligned}\dot{r} &= -\sin\phi\,\ddot{\theta}_d + \cos\phi\,\cos\theta\,\ddot{\psi}_d \\ &\quad -\cos\phi\,\dot{\phi}\,\dot{\theta} - \sin\phi\,\cos\theta\,\dot{\phi}\,\dot{\psi} - \cos\phi\,\sin\theta\,\dot{\theta}\,\dot{\psi}\end{aligned} \tag{43-3}$$

where $\ddot{\phi}_d, \ddot{\theta}_d, \ddot{\psi}_d$ and $\ddot{z}_d$ are given by (42-1) to (42-4) where appear the current target values for ϕ and θ, ϕ_{cc} and θ_{cc}, and the final target values of ψ and z, ψ_c and z_c.

5.2 Design of horizontal guidance control law

Now, considering equations (35-1) and (35-2), to insure that x and y adopt second order dynamics such as:

$$\ddot{x} + 2\,\zeta_x\,\omega_x\,\dot{x} + \omega_x^{\,2}(x - x_c) = 0 \tag{44-1}$$

$$\ddot{y} + 2\,\zeta_y\,\omega_y\,\dot{y} + \omega_y^{\,2}(y - y_c) = 0 \tag{44-2}$$

following the non linear inverse control approach, ϕ_{cc} and θ_{cc} must be chosen such as:

$$\begin{aligned}&(1/m)((\cos\psi\,\sin\theta_{cc}\,\cos\phi_{cc} + \sin\psi\,\sin\phi_{cc})\,u_z) \\ &\quad + 2\,\zeta_x\,\omega_x\,\dot{x} + \omega_x^2(x - x_c) = 0\end{aligned} \tag{45-1}$$

$$\begin{aligned}&(1/m)((\sin\psi\,\sin(\theta_{cc})\cos\phi_{cc} - \cos\psi\,\sin\phi_{cc})\,u_z) \\ &\quad + 2\,\zeta_y\,\omega_y\,\dot{y} + \omega_y^{\,2}(y - y_c) = 0\end{aligned} \tag{45-2}$$

Then :

$$\phi_{cc} = \arcsin(m(\sin\psi\;D_x - \cos\psi\;D_y)/u_z) \tag{46-1}$$

$$\theta_{cc} = \arcsin(m(\cos\psi\;D_x + \sin\psi\;D_y)/(u_z\cos\phi_c)) \tag{46-2}$$

where

$$D_x = -2\,\zeta_x\,\omega_x\,\dot{x} - \omega_x^{\,2}(x - x_c) \tag{47-1}$$

$$D_y = -2\,\zeta_y\,\omega_y\,\dot{y} - \omega_y^{\,2}(y - y_c) \tag{47-2}$$

5.3 Cases studies

Here we consider two cases: one where the objective is to hover at an initial position of coordinates x_0, y_0, z_0 while acquiring a new orientation ψ_1, and one where the rotorcraft is tracking the helicoïdal trajectory of equations:

$$x_c(t) = \rho\cos\nu\,t \tag{48-1}$$

$$y_c(t) = \rho\sin\nu\,t \tag{48-2}$$

$$z_c = \delta + \gamma\,t \tag{48-3}$$

$$\psi_c(t) = \nu\,t + \pi/2 \tag{48-4}$$

where ρ is a constant radius and γ is a constant path angle.

Heading control at hover

In this case we get the guidance control laws:

$$u_\psi = \frac{I_{zz}}{k}\ddot{\psi}_c \qquad u_z = m\,g \tag{49}$$

with the following reference values for the attitude angles:

$$\theta_c = 0 \quad \text{and} \quad \phi_c = 0 \tag{50}$$

Here the heading acceleration is given by:

$$\ddot{\psi}_c = -2\,\zeta_\psi\,\omega_\psi r - \omega_\psi^{\,2}(\psi - \psi_1) \tag{51}$$

Starting from an horizontal attitude ($\theta(0)$=0, $\phi(0)$=0), attitude inputs u_q and u_p remain equal to zero. Then, figures 4 and 5 display some correspondent simulation results.

Trajectory tracking case

In this case we get the guidance control laws:

$$u_\psi = 0 \qquad u_z = m\sqrt{\rho^2\nu^2 + g^2} \tag{52}$$

Here the permanent reference values for the attitude angles are such as:

$$\theta_c = 0 \tag{53}$$

and

$$\sin\phi_c = -\frac{\rho\,\nu^2}{\sqrt{\rho^2\nu^4 + g^2}} \tag{54}$$

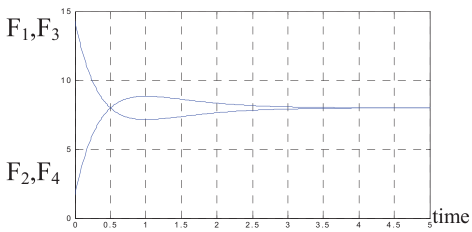

Fig. 4: Hover control inputs

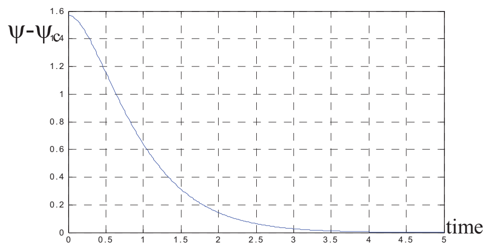

Fig. 5: Heading response during hover

and the desired guidance and orientation accelerations are given by:

$$\left.\begin{array}{c} \ddot{x}_c = -\rho\,\nu^2 \cos(\nu\, t) \\ \ddot{y}_c = -\rho\,\nu^2 \sin(\nu\, t) \\ \ddot{z}_c = 0, \quad \ddot{\psi}_c = 0 \end{array}\right\} \qquad (550)$$

In figures 6 to 8 simulation results are displayed where at

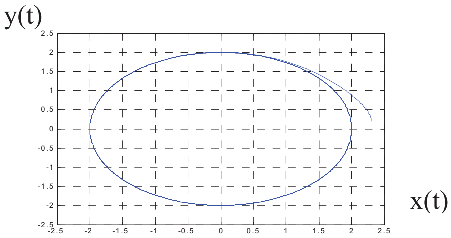

Fig. 6: Evolution of rotorcraft horizontal track

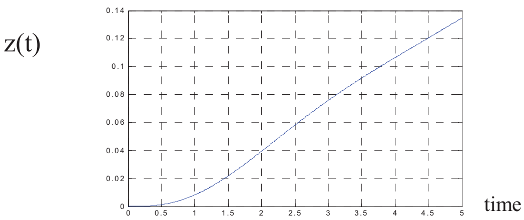

Fig. 7: Evolution of rotorcraft altitude

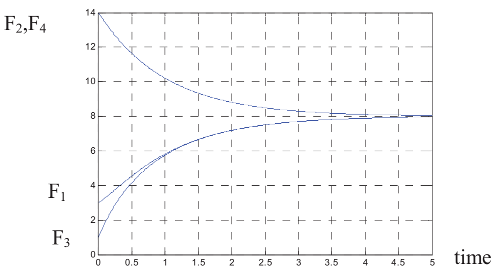

Fig. 8: Rotorcraft trajectory tracking inputs

initial time the rotorcraft is hovering.

6 Conclusion

In this communication the relation between differential flatness and the effective applicability of non linear inverse control approach has been studied. Considering the relative degrees of the outputs of a non linear system, necessary and sufficient conditions for differential flatness have been displayed. It has been shown that the application of the non linear inverse approach to an output observable and output differential flat system leads to an output feedback control law identical to the one derived from the differential flat control approach. Then the non linear flight dynamics of a rotorcraft have been analyzed and it has been shown that these dynamics are differential flat with output observability and output controllability properties. The application of the non linear inverse control approach to this four rotor aircraft has been considered and a trajectory tracking control structure based on two non linear inverse control layers has been proposed.

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