Using lotteries in auctions when buyers collude

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Abstract

This paper studies the optimal auction for a seller who is bound to sell a single item to one of two potential buyers organized in a "well-coordinated" cartel. After discussing the way the cartel reacts to any auction mechanism, we show that if the seller has no way to deter collusion, he can still accomodate it optimally with a very simple mechanism, either having the cartel pay to get an efficient allocation or randomly allocating the item. We then discuss the way to implement this mechanism, so that it enables a fair amount of competition if the seller made a mistake and the buyers don't collude. We find that a simple implementation using reserve prices and lotteries may yield expected revenues close to the optimum if buyers compete, while highly increasing expected revenues if they collude. Finally, we discuss the extension to the n-buyers case.

Key words: collusion, cartel, optimal auction, mechanism design..

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1 Introduction

Many authors have reported evidence of collusion in auctions.[1] This is clearly a major concern for the seller, even if he is a government and is mainly concerned with efficiency: Indeed, given a certain level of government spending, those revenues which are raised through auctions don't have to be raised through efficiency distorting taxes. Moreover, when the cartel can not use transfers among its members, collusion sometimes takes the form of either randomizing who will get the auctioned item, or choosing the winner by rotating among the cartel members, even further degrading efficiency.

In some cases, the seller can deter collusion by choosing an appropriate auction format: For example, if there are few further interaction between the buyers after the auction, and side-payments between them are not possible, the seller can reasonably use a sealed-bid auction to sell the item (for this type of auction, cartels are unstable, as even if an agreement is met between the buyers prior to the auction, some of them will have incentives to deviate from this agreement once the auction runs, and other members of a the cartel will not be able to react to this deviation[2]). However, this is often not possible: If buyers are patient enough and are engaged in a repeated collusion/competition game with uncertain horizon involving the seller relatively rarely, there is little he can do to design his auction in order to upset a grim trigger strategy equilibrium.

Still, the seller can accommodate collusion, and is usually advised to use high reserve prices to limit its adverse effects. Yet, this is frequently problematic when the government has to allocate essential facilities, needed for a valuable service to be offered to consumers, or wants to procure something it values much[3] (defence procurement for example): In many cases, it is thus clear that the government will sell the item, even at a very low price, so that no reserve price will appear as credible. Then, it might seem that the seller has no way to increase his revenue above the minimum possible buyer's valuation, which we will show not to be true.

[1] See Baldwin, Marshall, and Richard [5], Comanor and Schankerman [7], Cook [8], Graham and Marshall [11], Hay and Kelley [12], Marshall and Meurer [15], Pesendorfer [21], Porter and Zona [23], Phillips, Menkhaus and Coatney [22], Mund [18], Stigler [26].

[2] This is not the case in "open" auctions: If a cartel member decides to deviate from the cartel's assignment, other cartel members will see it and have the opportunity to react, thus lowering what the deviating member gains from deviating. In this sense, open auctions make collusion easier to sustain.

[3] Another argument, is that not selling the item can be perceived as a failure of the government, which may appear politically unacceptable to it.

In this paper, we will focus on the case where the seller of a single item has no way to break a cartel and is bound to sell the item, and will look for the best way for the seller to accommodate collusion.

This clearly depends on the way the cartel is organized. After presenting the model in section 2, we'll discuss this issue in section 3. We'll see that when the cartel members can use side payments between them or can communicate before the auction, it is reasonable to assume that, whatever the auction, the cartel will manage to achieve the maximum efficiency attainable to him. Taking this into account, in section 4 we'll derive the optimal auction mechanism, when the cartel is composed of two potential buyers and the seller can prevent the resale of the item: We'll show that the seller's revenue can be increased by having the cartel choose to either randomly draw which buyer gets the item at the minimum price or have one buyer get the item for sure, but at a higher price. In section 5, we'll show how to implement this auction mechanism, in a way so that a "fair amount" of competition occurs in case the seller made a mistake and buyers don't collude. We propose to use a reserve price auction, and to randomly allocate the item for the minimum price in case no bid exceeds the reserve price. We then derive the equilibrium strategies for this implementation if the buyers compete, and find that this auction may not yield much lower revenues than a classical auction. Finally, we extend this implementation to a n-buyers auction in section 6 and propose an auction where a lottery determines the winner among the buyers placing the highest k bids, for a price equal to the $k-th$ bid, where the "size" of the lottery k is all the larger as more bids are low. Section 7 discusses other ways to fight collusion and concludes.

2 The model

A seller has to sell an item through an **auction mechanism** he can design. We assume he is willing to sell the item whatever revenue he gets from it, and can prevent the resale of the item.

There are two potential buyers $B1$ and $B2$.

We denote v_1 and v_2, buyer $B1$ and buyer $B2$'s private valuations for the item, assumed to be drawn from a given symmetric continuous joint density function,[4] whose support is $S = [\underline{v}, \overline{v}]^2$. The valuation v_i is only known by buyer Bi. These valuations' marginal density function is f_m, their distribution function is F_m. Buyers are risk-neutral.

We assume that the buyers are part of a cartel, which manages to coordinate the buyers' actions so that the cartel's ex-post total surplus is the maximum attainable to him for the given auction mechanism (we discuss this assumption in the next section, and show that it is not unrealistic provided the cartel can use present or future transfers between its members).

Let $d = v_1 - v_2$ be the difference between the two buyers' valuations, f_Δ its density function on $S_\Delta = [\underline{v} - \overline{v}, \overline{v} - \underline{v}]$, and F_Δ its distribution function.

All of this is common knowledge.

Finally, we assume that the function J, defined by $J(d) = d - \frac{1-F_\Delta(d)}{f_\Delta(d)}$ is non-decreasing on $[0, \overline{v} - \underline{v}]$. This is a classical assumption in auction theory and is verified with most standard distribution functions F_Δ[5].

3 The efficiency of the cartel

In the model, we assumed that the cartel manages to get ex-post the maximum surplus it could reach. This is a very important assumption, since it determines what auction mechanism maximizes the sellers' revenue. In this section, reviewing part of the literature on cartels, we explain why this assumption is not so unrealistic when a cartel can make present or future side payments between its members.

Suppose buyers organised in a cartel can participate an auction mechanism. Following the literature, we consider a third party, the cartel, can design a **collusion mechanism** to assign their bids to the cartel members. The cartel's goal is to maximize ex ante the sum of its members' surplus (i.e. before buyers' valuations are known). Here, we assume that the buyers have exogenous incentives to participate the cartel and not to deviate from the actions which are assigned to them as a result of

[4] Note that the valuations are not necessarily independant.

[5] One could wonder whether this is still verified for distribution functions F_Δ deriving from most standard distributions of the valuations. Yet, we could relax this assumption, by only assuming that J equals 0 only for a finite number of values, but this would make the exposition of our results more complex without giving more insight into our problem.

the collusion mechanism (for example because deviating from this action would trigger a competition war in future interactions): So, all that is required from the mechanism is to give incentives to the cartel members to truthfully declare their valuation for the item.

In their paper "Bidding Rings", Mc Afee and Mc Millan [14] have been the first to study this problem, focusing on English auctions with reserve price, for which they analyze the optimal collusion mechanism (i.e. the mechanism maximizing ex-ante the cartel members' total surplus, in a setting where the cartel members' valuations for the auctioned item are private, independent and identically distributed). They consider two cases: **strong cartels** and **weak cartels** according to whether side payments are possible or not.

3.1 The strong cartel (with side payments)

Mc Afee and Mc Millan [14] show that if the cartel is strong, one of the optimal collusion mechanisms for the cartel is to have a prior auction to decide who will get the item at the reserve price r: All the cartel members have to declare how much they agree to pay the cartel for it, and eventually this payment is split equally between all the non-winning members. This budget-balanced collusion mechanism achieves efficiency as the buyer with the highest valuation always wins the prior auction, as soon as its valuation exceeds r.

In fact, if the buyers' valuations are independant, this can be generalized to any auction mechanism the seller chooses (provided the item has to be allocated), just by using a D'Aspremont/Gerard-Varet [1] collusion mechanism (hereafter referred to as AGV mechanisms):

To see that, imagine a seller has chosen an auction mechanism. If the cartel members choose actions in this auction which give the item to buyer $B1$ with probability P_1 (and thus to buyer $B2$ with probability $P_2 = 1 - P_1$) for payments by cartel members equal to T_1 and T_2, then the cartel members' total surplus is

$$v_1 P_1 + v_2 P_2 - T_1 - T_2 = (v_1 - v_2) P_1 - T_1 - T_2 + v_2.$$

So, for a given difference $d = (v_1 - v_2)$ between buyer $B1$ and buyer $B2$'s valuations, it is the same set of actions that maximize the cartel members' total surplus. Let's denote $P_i(d)$ and $T_i(d)$ the outcome of one of these actions maximizing cartel surplus.

Then, in order to have its members truthfully reveal their valuations, the cartel can use the following AGV collusion mechanism: If buyer $B1$ announces $\widehat{v}_1$, and buyer $B2$ announces $\widehat{v}_2$, the buyers are assigned to choose their actions in the auction mechanism so as to get the outcome $(P_i(\widehat{v}_1 - \widehat{v}_2)\ ,\ T_i(\widehat{v}_1 - \widehat{v}_2))$ and intra-cartel transfers are made according to the following rules:

$$t_1(\widehat{v}_1, \widehat{v}_2) = E_{v_1}\left[v_1 P_1(v_1 - \widehat{v}_2) - T_1\left(v_1 - \widehat{v}_2\right)\right] - E_{v_2}\left[v_2 P_2(\widehat{v}_1 - v_2) - T_2\left(\widehat{v}_1 - v_2\right)\right] = -t_2(\widehat{v}_1, \widehat{v}_2)$$

This is a classical AGV mechanism, and it is easy to check[6] that truth-telling constitutes a Bayesian-Nash equilibrium.

Of course, there might be other budget-balanced collusion mechanism, yielding the same cartel's ex-ante expected surplus, but they would all imply that the cartel perfectly manages to coordinate the buyer's actions in order to get the maximum cartel's surplus attainable ex-post with the given auction mechanism.

So the seller knows that whatever auction mechanism he chooses, he must expect the cartel members' actions to be the same as if he was facing a single entity: the cartel.

3.2 The weak cartel (no side payments)

If the cartel is weak, Mc Afee and Mc Millan [14] show[7] for an English auction, that an optimal collusion mechanism, such that cartel members are always treated symmetrically, is to randomly decide who will be the winner among those members whose valuation exceed the reserve price r. A classical variant of this mechanism is to choose the winner by rotating among the cartel members (rotating bids). Another variant is to let the members decide to place a bid equal to r or below, and let the auctioneer do the randomizing.

[6] The first part of $t_1(\widehat{v}_1, \widehat{v}_2)$ does not depend on buyer $B1$'s announce $\widehat{v}_1$. So, if buyer $B2$ announces it true value v_2, buyer $B1$ wants to maximise

$$\begin{aligned} & E_{v_2}\left[v_1 P_1(\widehat{v}_1 - v_2) - T_1\left(\widehat{v}_1 - v_2\right)\right] + E_{v_2}\left[v_2 P_2(\widehat{v}_1 - v_2) - T_2\left(\widehat{v}_1 - v_2\right)\right] \\ = \; & E_{v_2}\left[v_1 P_1(\widehat{v}_1 - v_2) + v_2 P_2(\widehat{v}_1 - v_2) - T_1\left(\widehat{v}_1 - v_2\right) - T_2\left(\widehat{v}_1 - v_2\right)\right]. \end{aligned}$$

By definition of P_i and T_i, whatever B2's valuation v_2 , the term $v_1 P_1(\widehat{v}_1 - v_2) + v_2 P_2(\widehat{v}_1 - v_2) - T_1\left(\widehat{v}_1 - v_2\right) - T_2\left(\widehat{v}_1 - v_2\right)$ is maximum for $\widehat{v}_1 - v_2 = v_1 - v_2$,that is for $\widehat{v}_1 = v_1$. So truthtelling constitutes a Bayesian-Nash equilibrium.

[7] under a classical technical assumption on the valuations' distribution which is verified for most distribution. Namely, that $H(v) = \frac{1-F(v)}{f(v)}$ is non-increasing.

Without the possibility to do transfers among its members, a weak cartel therefore performs very poorly in capturing the efficiency available to him if he **always** treats his member symmetrically..

Yet, recent research has shown that weak cartels can improve efficiency if its members frequently interact, by not always treating the cartel members identically. Fudenberg, Levine and Maskin [10] have extended the Folk Theorem to repeated adverse selection models with independent private values: Applying their theorem shows that if the cartel members communicate and are patient enough, the cartel is able to achieve efficiency without side-payments[8]. The idea is to keep an account of who wins, and to put former winners at a disadvantage, so that expected future surplus difference play the same role as side payments in AGV mechanisms.

Even if the cartel members don't communicate, Skrzypacz and Hopenhayn [25], and Blume and Heidues [6] have shown that while the cartel can't extract all the surplus he could theoretically get, he can still approach it: One example of collusion mechanism is to let the cartel members bid as they want, and to exclude the winner of an auction of a next few auctions, with a given probability. Another example is to keep an account of how many times each member has won and exclude one buyer from a few further auctions if this account becomes too dissymmetric. By artificially lowering the expected surplus a buyer gets from winning an auction (and transferring this loss to the losers), these collusion mechanism give buyers incentives to place lower bids, while still enabling a better allocation efficiency.

This shows, that the distinction between weak and strong cartel may not be the relevant one from the seller's point of view, or that we should consider side-payments in a larger sense, possibly integrating future pay-off transfers.

Thus, if cartel members can make side-payments or can communicate and frequently interact, one can reasonably assume that, whatever the auction mechanism, the cartel sets a collusion mechanism such that its surplus will be maximized ex-post.

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[8] Aoyagi [3] also identifies conditions under which there exists an optimal collusion mechanism when the cartel members' valuations are not private. See Athey and Bagwell [4] for a related application to collusion in cournot competition.

4 An optimal auction mechanism when two buyers collude and the seller is bound to sell the item

In this section, consistently with our assumptions and with what we saw in the previous section, we take for granted that, whatever the auction proposed by the seller, the cartel can extract the full surplus attainable to him.

What can the seller do to improve his revenues if he bound to sell an item while facing a single entity maximizing his surplus? To ensure that the item can be sold whatever the buyers' valuations, the seller has to make it possible to get the item at the lowest possible valuation $\underline{v}$. But, as the seller can prevent the resale of the item, there remains one thing he can control, and for which the cartel may be willing to pay an extra price: the fact that the item is allocated to the buyer who values it the most.

Indeed an optimal auction mechanism consists in proposing the cartel to either "buy" an efficient allocation (that is, the possibility to choose the buyer who gets the item) or to randomize the winner of the auction.

Proposition 1 *There exists a price r , with $\underline{v} < r < \underline{v} + \frac{\overline{v}-\underline{v}}{2}$, such that one optimal auction mechanism is to propose two possible outcomes:*

- *either randomly allocate the item to one of the two buyers (with equal chance) at price $\underline{v}$,*
- *or let the cartel choose who gets the item at a higher price r.*

  This price r is such that $2(r - \underline{v})$ is the unique solution to

$$x = \frac{1 - F_{\Delta}(x)}{f_{\Delta}(x)}.$$

  This leads the item to be allocated randomly when the difference between the buyers valuations is inferior to $2(r - \underline{v})$, and to the highest valuing buyer otherwise

Proof: Refer to Appendix [9] ■

[9] While the result is intuitive, the proof is not so obvious, and does not enable much interpretation.

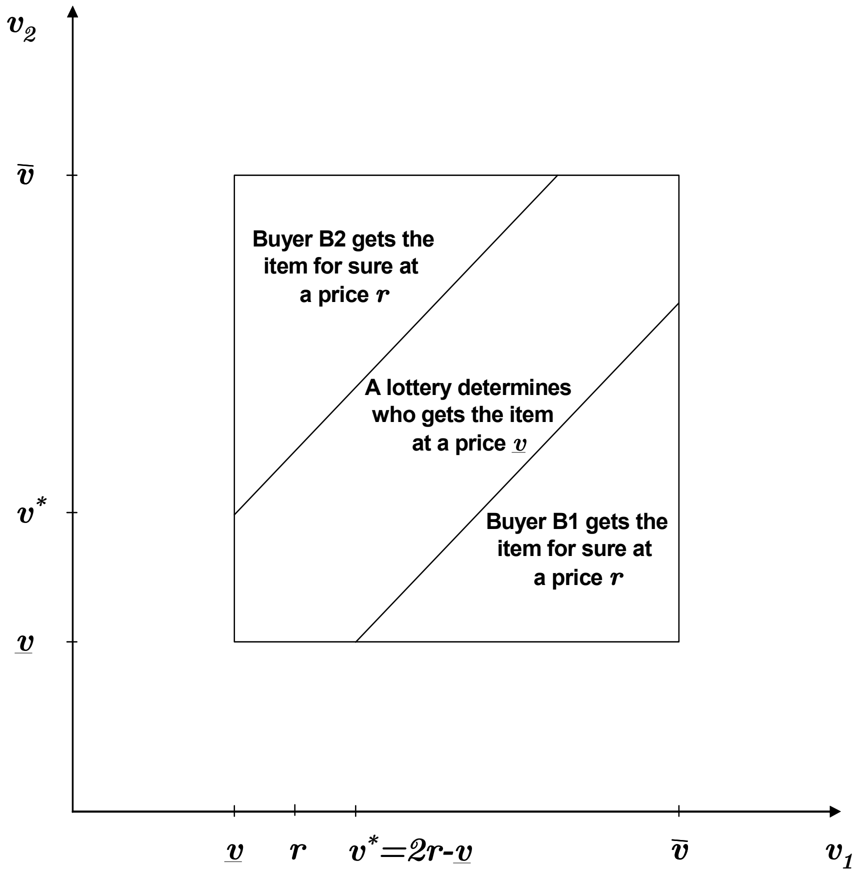


Figure 1: Outcome of the optimal mechanism described in proposition 1

By entangling the offer for which the item is sold at the lower price with the maximum possible allocation inefficiency (allocating the item randomly), the seller can charge a higher price in case the cartel chooses to benefit from an efficient allocation. This result can be related to the Mussa/Rosen model [19], by considering the efficiency of the allocation as a product's quality: The seller proposes two offers and chooses to degrade the quality of the offer intended to be chosen by a buyer paying little attention to quality, in order to have a buyer paying high attention to quality pay more for the efficient quality offer.

This result is quite intuitive, and might seem obvious, yet it is not trivial: A priori the optimal auction mechanism might have consisted in a complex list of proposed random allocations with different corresponding payments. Surprisingly, whatever the valuations' distribution, offering the above two possible outcomes is enough to maximize the seller's revenue.

Note that to implement this mechanism, the seller does not have to address the cartel as an entity: A simple implementation would consist in asking the buyers to either bid $\underline{v}$ or a higher price r, informing them that in case of a tie, the winner is randomly drawn. This raises the participation issue: what if a buyer declares that he doesn't want to participate the auction? This could be a way for the cartel to choose the buyer getting the item while paying the minimum price $\underline{v}$. In order to prevent that, the seller has to commit not to sell the item if any of the two buyers does not participate, unless the other buyer agrees to pay the higher price r. Knowing that, if the commitment is credible, both buyers will participate and the seller does not take any risk of not selling the item.

Now, it is clear that in case the seller made a mistake and a cartel does not exist, this mechanism would not perform very well, as the seller's revenue would never exceed the price r. The problem is how to design an auction such that, if there is a cartel, he is faced with the optimal mechanism described above, and if the buyers compete, they have incentives to place bids that may result in high payments. This is the subject of the next section.

5 Implementation of this optimal auction mechanism.

Another and better way to implement this mechanism, so as to introduce more competition if buyers do not collude, would be to conduct a "normal" auction with reserve price r and, in case no buyer places a valid bid, to have a lottery determine the winner at a price $\underline{v}$.[10] Furthermore, the seller commits not to sell the item at a price below the reserve price r if one of the buyers does not participate the auction mechanism.

With this auction, the competing buyer's equilibrium strategies differ from those of classical auctions with reserve price, as their expected surplus when they do not place a valid bid depends on their valuation (this is because if they don't place a valid bid, they still have a chance to get the item thanks to the lottery).

If the buyers compete and have independant identically-distributed valuations, whatever auction format is chosen among the four classical auctions (English, Dutch, first-price sealed-bid, second-price sealed bid) to implement the optimal mechanism in case of collusion, there exists a Bayesian-Nash equilibrium yielding the following allocation: If both buyers have valuations below **the threshold** $v^* = 2r - \underline{v}$ ($v^* \leq r$), a lottery takes place to determine the winner. If at least one buyer has a valuation above v^*, the buyer with the highest valuation gets the item. (In fact, buyers only place bids above the reserve price when their valuations exceed the threshold v^*). We show this for English, and sealed-bid second-price auction type, and we give the idea of the proof for the sealed bid first price (or strategically-equivalent Dutch) auction in Appendix 2.

So, part of the buyers who might find it profitable to get the item at the reserve price r, prefer taking their chance to get the item at the lowest price $\underline{v}$ through the lottery, rather than participating the auction with reserve price. However, if one buyer chooses to participate the auction with reserve price r, his strategy is a classical competitive one, and the buyer who values the item the most gets it. As a consequence of the revenue equivalence theorem (Myerson [20]), the seller's expected revenue is the same for the four auctions formats (since revenue will always be equal to $\underline{v}$ when both buyers

[10] The English auction just needs to be slightly modified to enable buyers to continue bidding if all other buyers have dropped the auction.

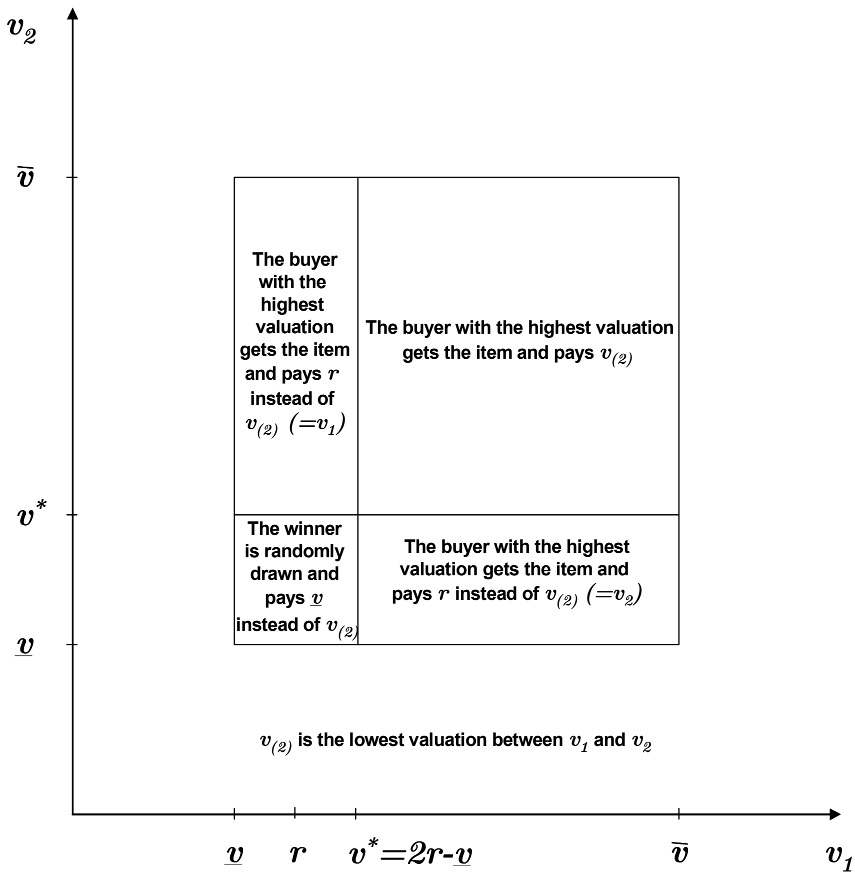


Figure 2: Allocative outcome of an auction with activity-dependant reserve price and lottery when two buyers compete. Ex-post revenue comparison with an auction with no reserve price for English or sealed-bid second-price auctions.

have the lowest valuation $\underline{v}$).

This enables us to evaluate the cost of choosing an auction with activity-dependant reserve price and lottery, if buyers compete. As an example, if the buyers' valuations are distributed uniformly on $[\underline{v}, \underline{v}+l]$, one can easily compute that expected revenue when two buyers compete is $\underline{v}+l/3$ for any of the four classical auctions without reserve price and $\underline{v}+26/27*l/3$ if the optimal anti-collusion mechanism is implemented via activity-dependant reserve price and lottery, while the expected revenue if there is a strong cartel goes from $\underline{v}$ to $\underline{v}+2/9*l/3$.

In fact, while expected revenue is always lower with the auction with activity-dependant reserve price and lottery than with classical auctions, for some particular valuations the ex-post revenue is actually higher (figure 2). For example, for a second price or English auction, this is the case when one buyer has a valuation above v^* while the other buyer's valuation is below r : In this case, the reserve price plus lottery auction yields a revenue equal to r instead of a revenue equal to the lowest valuation $v_{(2)}$.

When both buyers have valuations above v^*, the revenues are the same (the second highest valuation). When both buyers have valuations below v^*, the auction with activity-dependant reserve price and lottery yields a revenue equal to $\underline{v}$ instead of the lowest valuation $v_{(2)}$.

It can be shown that for a given r (determining the value of $v^* = 2r - \underline{v} = \underline{v} + 2(r - \underline{v})$), the expected revenue loss ΔR from adopting the auction with activity-dependant reserve price and lottery rather than classical auctions without reserve price is

$$\begin{aligned}
\Delta R(v^*) &= \int_{\underline{v}}^{v^*} (1 - F_m(t))^2 dt - (1 - F_m(v^*))(v^* - \underline{v}) \\
&\leq (v^* - \underline{v}) - (1 - F_m(v^*))(v^* - \underline{v}) \\
&\leq F_m(v^*)(v^* - \underline{v}).
\end{aligned}$$

So, for values of r near $\underline{v}$, corresponding to $(v^* - \underline{v})$ close to 0, the order of the expected revenue loss is at most of the second degree $(v^* - \underline{v})^2$ (while the order of revenue gain when there is a strong cartel is of the first degree $(v^* - \underline{v})$). In situations where the seller is uncertain whether or not he faces a strong cartel, it might thus be interesting to set a reserve price under which a lottery occurs (at least a small one).

6 The n-buyers case. Auctions with activity-dependant reserve prices and lotteries

We haven't been able to find the optimal mechanism with more than two buyers.

Yet, we saw that the 2-buyers problem is equivalent to finding the optimal way to sell the possibility to modify the allocation away from a low-price allocation to the cartel. The solution presented above can be seen as organizing a lottery to choose the winner and proposing the cartel to pay for excluding one buyer from the lottery

Thus, a natural extension of this mechanism in case of n potential buyers, would be to have a random draw to determine the winner of the auction, and to offer the cartel to pay to exclude one or more buyers from this lottery. It would then consist in n "reserve"-prices: $r_n = \underline{v} < r_{n-1} < r_{n-2} < .... < r_2 < r_1$, where r_i is the price to be paid for the item if the cartel decides that only i buyers participate the lottery.

In fact, there are some hints which suggest that such an auction mechanism might well be optimal.[11]

In order to implement that, so as to enable competition if buyers don't collude, we can just slightly modify a normal auction: First, a classical auction is conducted. Then, we look for the highest "reserve"-price r_k such that more than k buyers have submitted valid bids for this reserve-price. If no reserve-price r_k satisfies this condition, the item is not sold[12]. Otherwise, a lottery determines who gets the item among the k buyers who have submitted the highest bids, for the price that would have paid the buyer with the $k-th$ highest bid in the classical auction.

For example, if there are three potential buyers and $r_3 = 1$, $r_2 = 3$, $r_1 = 4$. If the bids placed through a first price sealed-bid auction are 1.5, 3.2 and 4.1, the bid 4.1 wins and the winner pays 4.1 for the item. If the bids are 1.5, 3.2 and 3.9, a lottery is conducted between the two highest-bid buyers and the winner pays 3.2. If the bids are 1.5, 2.9 and 3.9, a lottery is conducted between the

[11] If, as in the 2-buyers case, there exists an optimal aution mechanism which never uses a lottery, then one of this lottery-type auction mechanism is also optimal. This can be shown by symmetrising the deterministic mechanism.

[12] This is just to ensure that a cartel could not achieve to choose whose buyer gets the item while paying the minimum price $\underline{v}$. As $r_n = \underline{v}$, the item will always be sold.

three buyers and the winner pays 1.5.

In fact, this could also be implemented sequentially. A first auction with reserve price r_1 is run. If no buyer participates this auction, then a second auction with a lower reserve price r_2 is run, but the buyers are told that the item will be sold via a lottery to one of the two highest bidders, except if less than two bids are valid. In this case, a third auction with a lower reserve price r_3 is run, etc. This is why we talk about activity-dependant reserve prices and lotteries.

There has already been auctions using reserve-price depending on the number of participating firms, but this isn't enough to improve revenues if all buyers collude: they could just all participate in order to meet the number of participating firms triggering a low reserve price, and place very low bids. Adding lotteries, whose allocation inefficiency is higher when bids are low, is needed to give a cartel the correct incentive to place higher bids.

7 Conclusion

When collusion is suspected in auctions, the most recommended way to improve the seller's revenue is to set a high reserve price. Frequently, when the seller is a government, it is not possible as it is clear that it is willing to sell the item no matter what revenue he gets from it.

In this paper, we focused on the case where the seller has no way to deter collusion, for example because he has very few contacts with the cartel members as compared to the numerous interactions between them, and no entry from new buyers is possible. So we looked for the optimal way to accommodate collusion rather than fighting it. We saw that, if a seller is bound to sell a single item and faces a strong cartel (able to make side-payments[13] among his members) he can't break, he can still improve his revenues by setting a reserve price below which a lottery occurs, and can't do any better. This is in contrast with his situation when he faces a weak cartel whose members don't communicate. In this case, while the cartel can still achieve some coordination by using bid rotation schemes with temporary exclusion of past winners, he can't fully extract the surplus attainable. This may paradoxically not be good news for the seller, as it limits his possibility to have the cartel pay

[13] As we saw in section 3, "side-payment" has to be taken in its larger sense, possibly including future pay-off transfers.

for an efficient allocation.

Otherwise, a first way to fight collusion is to foster entry from new competitors. Still, there is the problem that the entrant could join the cartel, but it is classical that cartels are extremely difficult to maintain as the number of its members grow.

However, sometimes entry is not possible, at least in the short run. Even then, there remains other ways to try and break the cartel, provided the seller interacts often enough with the cartel members. What enforces a cartel is the fear that deviating from the agreement reached within the cartel, would trigger a competition war in further interactions. So the cartel's enforcement results from the trade-off between what the cartel members loose if competition occurs instead of collusion in the future, and what a cartel member can gain now if he deviates from the cartel's recommendation.

So, trivially, one can fight a cartel by trying to lower what cartel members gain from colluding, and to increase what they gain by competing in further rounds and by deviating from the cartel's recommendation now while others stick to them. Leniency programs and higher collusion control expenses do that. The mechanism we propose also does, by lowering collusion gains and increasing competition gains.

An interesting way to look for the optimal revenue generating mechanism when the cartel members frequently interact would be to combine both approaches: collusion accommodation and collusion deterrence. The idea would be to design an auction mechanism taking into account the possibility that for some valuations, buyers will choose to compete and break the cartel, and that for some other valuations they'll choose to collude. For example, one might design an auction so that buyers with high valuations have very high incentives to deviate, which, amplified by a snowball effect, could be enough to break the cartel (it could lead buyers with intermediate valuations to wonder whether it's interesting for them to participate a cartel whose members only collude when they have low valuations). Of course, this would depend on the collusion mechanism designed by the cartel, which could itself depend on the auction mechanism. This is a problem that seems extremely complex to deal with, but it might perhaps be done by restricting one's attention to a particular class of auctions. Auctions with activity-dependant reserve prices and lotteries, being a simple generalization of classical auctions, might be a good starting point to study this problem.

8 Appendix 1

Proof of proposition 1: As the cartel sets a mechanism inciting the buyers to perfectly coordinate their actions in order to maximize the total surplus, we can consider the seller faces a single agent: the cartel. The revelation principal (Myerson [20]) states that if there exists a Bayesian-Nash equilibrium for a particular mechanism, there exists a direct revelation mechanism[14] yielding the same expected revenue.

Thus, we will now restrict our attention to direct revelation mechanisms. We will denote $P_i(\widehat{v}_1, \widehat{v}_2)$ the probability that buyer Bi gets the item if the cartel announces a valuation $\widehat{v}_1$ for buyer 1 and a valuation $\widehat{v}_2$ for buyer 2. We'll use $T(\widehat{v}_1, \widehat{v}_2)$ as the corresponding expected payment (including both what buyer $B1$ and buyer $B2$ have to pay).

We're looking for the mechanism (P_1, P_2, T) which generates the highest expected revenue for the seller among those that satisfy the three constraints:

$$\begin{aligned}
\forall\left(\left(v_1, v_2\right), \left(\widehat{v}_1, \widehat{v}_2\right)\right) &\in S^2 \\
v_1 P_1\left(v_1, v_2\right) + v_2 P_2\left(v_1, v_2\right) - T\left(v_1, v_2\right) &\geq v_1 P_1\left(\widehat{v}_1, \widehat{v}_2\right) + v_2 P_2\left(\widehat{v}_1, \widehat{v}_2\right) - T\left(\widehat{v}_1, \widehat{v}_2\right) \text{ (IC)} \\
v_1 P_1\left(v_1, v_2\right) + v_2 P_2\left(v_1, v_2\right) - T\left(v_1, v_2\right) &\geq 0 \text{ (IR)} \\
P_1\left(v_1, v_2\right) + P_2\left(v_1, v_2\right) &= 1 \text{ (allocation).}
\end{aligned}$$

The (IC) constraint means that the agent has no better strategy than announcing the true valuations. The (IR) constraint means that whatever their valuations, the cartel will be willing to participate. The (allocation) constraint is necessary to ensure that the item will be sold whatever the buyers' valuations.

In the first steps of the proof, we'll work on these constraints to show that this problem can be rewritten as a one-dimensional problem. Then, following Myerson's methodology ([20]), we'll express the expected revenue as a function of the allocation rules P_1 and P_2, and solve the problem.

[14] that is, a mechanism asking the agents to reveal their values, such that truthfully revealing those values constitutes a Bayesian-Nash equilibrium.

- Step 1 : Shows that if the (IR) constraint holds for the minimum valuations $v_1 = v_2 = \underline{v}$, then it holds for any v_1 and v_2.

Let v_1 and v_2 be given. The (IC) constraint implies that

$$v_1 P_1 (v_1, v_2) + v_2 P_2 (v_1, v_2) - T (v_1, v_2) \geq v_1 P_1 (\underline{v}, \underline{v}) + v_2 P_2 (\underline{v}, \underline{v}) - T (\underline{v}, \underline{v}).$$

As $P_1 (\underline{v}, \underline{v})$ and $P_2 (\underline{v}, \underline{v})$ are positive and $v_i \geq \underline{v}$, then

$$v_1 P_1 (\underline{v}, \underline{v}) + v_2 P_2 (\underline{v}, \underline{v}) - T (\underline{v}, \underline{v}) \geq \underline{v} P_1 (\underline{v}, \underline{v}) + \underline{v} P_2 (\underline{v}, \underline{v}) - T (\underline{v}, \underline{v}),$$

so the (IC) constraints imply

$$v_1 P_1 (v_1, v_2) + v_2 P_2 (v_1, v_2) - T (v_1, v_2) \geq \underline{v} P_1 (\underline{v}, \underline{v}) + \underline{v} P_2 (\underline{v}, \underline{v}) - T (\underline{v}, \underline{v}).$$

This shows that if the (IC) constraint holds everywhere, the (IR) constraint holds everywhere if and only if it holds for $v_1 = v_2 = \underline{v}$.

Now for a mechanism to be optimal, this constraint must be saturated at this point (otherwise the seller could slightly increase the payment T for every types of valuations without upsetting the (IC) and (allocation) constraints).

So, we can restrict our attention to mechanisms for which

$$\underline{v} P_1 (\underline{v}, \underline{v}) + \underline{v} P_2 (\underline{v}, \underline{v}) - T (\underline{v}, \underline{v}) = 0,$$

or equivalently

$$T (\underline{v}, \underline{v}) = \underline{v}, \tag{1}$$

and don't pay anymore attention to the (IR) constraint.

- Step 2: Shows that we can restrict our attention to direct revelation mechanisms which have the same allocation and payment rules everywhere inside any given line $v_1 - v_2 = d$.

Using the allocation constraint, the (IC) constraint implies that for any (v_1, v_2) and $(\widehat{v}_1, \widehat{v}_2)$ belonging to S,

$$\begin{aligned} &(v_1 - v_2) P_1 (v_1, v_2) - T (v_1, v_2) + v_2 \\ &\qquad \geq (v_1 - v_2) P_1 (\widehat{v}_1, \widehat{v}_2) - T (\widehat{v}_1, \widehat{v}_2) + v_2, \end{aligned}$$

or, equivalently

$$\begin{aligned}&(v_1 - v_2)\, P_1(v_1, v_2) - T(v_1, v_2)\\&\qquad\qquad \geq (v_1 - v_2)\, P_1(\widehat{v}_1, \widehat{v}_2) - T(\widehat{v}_1, \widehat{v}_2)\,.\end{aligned}$$

This inequality must also hold if we inverse v and $\widehat{v}$, which gives

$$(\widehat{v}_1 - \widehat{v}_2)\, P_1(\widehat{v}_1, \widehat{v}_2) - T(\widehat{v}_1, \widehat{v}_2) \geq (\widehat{v}_1 - \widehat{v}_2)\, P_1(v_1, v_2) - T(v_1, v_2)\,.$$

Combining these two inequalities implies that,

$$\left[\begin{array}{c}\text{if } v_1 - v_2 = \widehat{v}_1 - \widehat{v}_2, \text{ then}\\ (v_1 - v_2)\, P_1(v_1, v_2) - T(v_1, v_2) \geq (v_1 - v_2)\, P_1(\widehat{v}_1, \widehat{v}_2) - T(\widehat{v}_1, \widehat{v}_2) \geq (v_1 - v_2)\, P_1(v_1, v_2) - T(v_1, v_2)\\ \text{so that } (v_1 - v_2)\, P_1(v_1, v_2) - T(v_1, v_2) = (v_1 - v_2)\, P_1(\widehat{v}_1, \widehat{v}_2) - T(\widehat{v}_1, \widehat{v}_2)\,.\end{array}\right] \tag{2}$$

This shows that for a direct revelation mechanism, the cartel is indifferent between announcing its true valuations, or announcing other valuations as long as the difference between the valuations announced remain equal to the difference between its true valuations. In other words, the cartel is indifferent between the offer which is designed for him and offers which are designed for other types on the same line $v_1 - v_2 = cst$.

Of course, among the offers designed to be chosen by the different possible types of cartel on a line $v_1 - v_2 = cst$, the seller prefers the ones corresponding to the higher payments $T(v_1, v_2)$.

Let $T(d)$ be the higher limit of the $T(v_1, v_2)$ on the line $v_1 - v_2 = d$:

$$T(d) = \sup\{T(v_1, v_2)\,/ v_1 - v_2 = d\}\,.$$

Let $P_1(d)$ be such that for any (v_1, v_2) on the line $v_1 - v_2 = d$, we have

$$(v_1 - v_2)\, P_1(v_1, v_2) - T(v_1, v_2) = (v_1 - v_2)\, P_1(d) - T(d)\,.$$

Changing all offers designed for types on the line $v_1 - v_2 = d$, from $P_1(v_1, v_2)$ to $P_1(d)$ and from $T(v_1, v_2)$ to $T(d)$, increases the expected revenue generated by the mechanism and does not violate the (IC) constraint (Indeed , if the (IC) constraint holds for a sequence of $(\widehat{v}_1, \widehat{v}_2)$, it must also hold at the limit). Moreover, we saw that the (IR) constraint is equivalent to $T(\underline{v}, \underline{v}) = \underline{v}$, so this change of rules would not have any incidence on the validity of the (IR) constraint either.

This shows that we can restrict our attention to direct revelation mechanisms which have the same allocation and payment rules $(P_1(d), T(d))$ everywhere inside any given line $v_1 - v_2 = d$.

- Step 3: Proves that the IC constraints imply that P_1 must be increasing in d, and calculates the derivative of a function which is related to the cartel's expected surplus.

Now that we have converted our problem into a unidimensional one, we can apply Myerson's methodology ([20]).

Whatever d and $\widehat{d}$ belonging to S_Δ, the (IC) constraints imply

$$dP_1(d) - T(d) \geq dP_1\left(\widehat{d}\right) - T\left(\widehat{d}\right), \text{ (IC')}$$

and, inverting d and $\widehat{d}$,

$$\widehat{d}P_1\left(\widehat{d}\right) - T\left(\widehat{d}\right) \geq \widehat{d}P_1(d) - T(d).$$

Thus, we have

$$d(P_1\left(\widehat{d}\right) - P_1(d)) \leq T\left(\widehat{d}\right) - T(d) \leq \widehat{d}(P_1\left(\widehat{d}\right) - P_1(d)), \tag{3}$$

which implies

$$0 \leq \left(\widehat{d} - d\right)(P_1\left(\widehat{d}\right) - P_1(d)).$$

So, for the (IC) constraints to hold, P_1 has to be increasing in d.

Let's denote w the function defined by $w(d) = dP_1(d) - T(d)$. We now show that w is everywhere continuous, and differentiable except on an at most numerable number of points, with derivative P_1.

From the inequality 3,

$$dP_1(d) - \widehat{d}P_1\left(\widehat{d}\right) + d(P_1\left(\widehat{d}\right) - P_1(d)) \leq w(d) - w(\widehat{d}) \leq dP_1(d) - \widehat{d}(P_1\left(\widehat{d}\right) + \widehat{d}(P_1\left(\widehat{d}\right) - P_1(d)),$$

that is

$$(d - \widehat{d})P_1\left(\widehat{d}\right) \leq w(d) - w(\widehat{d}) \leq (d - \widehat{d})P_1(d),$$

which proves that w is continuous and differentiable at every d for which P_1 is continuous (everywhere except on at most numerable set of points, as P_1 is increasing): If $P_1(.)$ is continuous at a point d^*, then w is differentiable at d^*, and $w'(d^*) = P_1(d)$.

- Step 4: expresses the expected revenue with respect to the function $P_1(.)$.

As we restricted our attention to mechanisms which attributed the same payment to all types of agents having the same difference d, we can express the expected revenue by integrating over the values of d : So,

$$\begin{aligned}
E(T) &= \int_{S_\Delta} T(\delta) f_\Delta(\delta) d\delta \\
&= \int_{S_\Delta} (\delta P_1(\delta) - w(\delta)) f_\Delta(\delta) d\delta \\
&= \int_{S_\Delta} \delta P_1(\delta) f_\Delta(\delta) d\delta - \int_{S_\Delta} w(\delta) f_\Delta(\delta) d\delta.
\end{aligned}$$

As the function $\omega(.) F_\Delta(.)$ is continuous, differentiable everywhere except on an at most numerable set of points, it is a generalized primitive of the function $\omega(.) f_\Delta(.) + P_1(.) F_\Delta(.)$ and we can integrate by part the second integral. So,

$$\begin{aligned}
E(T) &= \int_{S_\Delta} \delta P_1(\delta) f_\Delta(\delta) d\delta - [w(\delta) F_\Delta(\delta)]_{\underline{v}-\overline{v}}^{\overline{v}-\underline{v}} + \int_{S_\Delta} P_1(\delta) F_\Delta(\delta) d\delta \\
&= \int_{S_\Delta} (\delta f_\Delta(\delta) + F_\Delta(\delta)) P_1(\delta) d\delta - w(\overline{v} - \underline{v}) \\
&= \int_{S_\Delta} (\delta f_\Delta(\delta) + F_\Delta(\delta)) P_1(\delta) d\delta - (w(\overline{v} - \underline{v}) - w(0)) - w(0) \\
&= \int_{S_\Delta} (\delta f_\Delta(\delta) + F_\Delta(\delta)) P_1(\delta) d\delta - \int_{\underline{v}-\underline{v}}^{\overline{d}} P_1(\delta) d\delta - w(0) \\
&= \int_{\underline{d}}^{\underline{v}-\underline{v}} (\delta f_\Delta(\delta) + F_\Delta(\delta)) P_1(\delta) d\delta + \int_{\underline{v}-\underline{v}}^{\overline{d}} (\delta f_\Delta(\delta) + F_\Delta(\delta) - 1) P_1(\delta) d\delta - w(0).
\end{aligned}$$

Now, in order to have an expression of expected revenue depending only on the function P_1, we have to use the fact that the (IR) constraint is saturated for the minimum valuations $v_1 = v_2 = \underline{v}$: Since from equality (1), $T(\underline{v}, \underline{v}) = \underline{v}$, we have

$$\begin{aligned}
w(0) &= w(\underline{v} - \underline{v}) \\
&= (\underline{v} - \underline{v}) P_1(\underline{v}, \underline{v}) - T(\underline{v}, \underline{v}) \\
&= -\underline{v}.
\end{aligned}$$

So

$$E(T) = \int_{\underline{d}}^{\underline{v}-\underline{v}} \left(\delta f_{\Delta}(\delta) + F_{\Delta}(\delta)\right) P_1(\delta)\, d\delta + \int_{\underline{v}-\underline{v}}^{\overline{d}} \left(\delta f_{\Delta}(\delta) + F_{\Delta}(\delta) - 1\right) P_1(\delta)\, d\delta + \underline{v}. \tag{4}$$

We will now look for the function P_1 which maximizes this expression, among the increasing functions taking their values in $[0,1]$. This will enable us to evaluate the corresponding payment function T. Finally, we'll check that the (IC) constraints hold for this solution.

- Step 5: Determines the optimal allocation.

Let L be the function defined by: $L \begin{cases} S_{\Delta} \longrightarrow R \\ d \longmapsto \begin{array}{ll} d + \frac{F_{\Delta}(d)}{f_{\Delta}(d)} & \text{if } d < 0 \\ d - \frac{1-F_{\Delta}(d)}{f_{\Delta}(d)} & \text{if } d \geq 0 \end{array} \end{cases}$, so that expected revenue can be expressed as

$$E(T) = \int_{S_{\Delta}} L(\delta)\, P_1(\delta)\, f_{\Delta}(\delta)\, d\delta + \underline{v}.$$

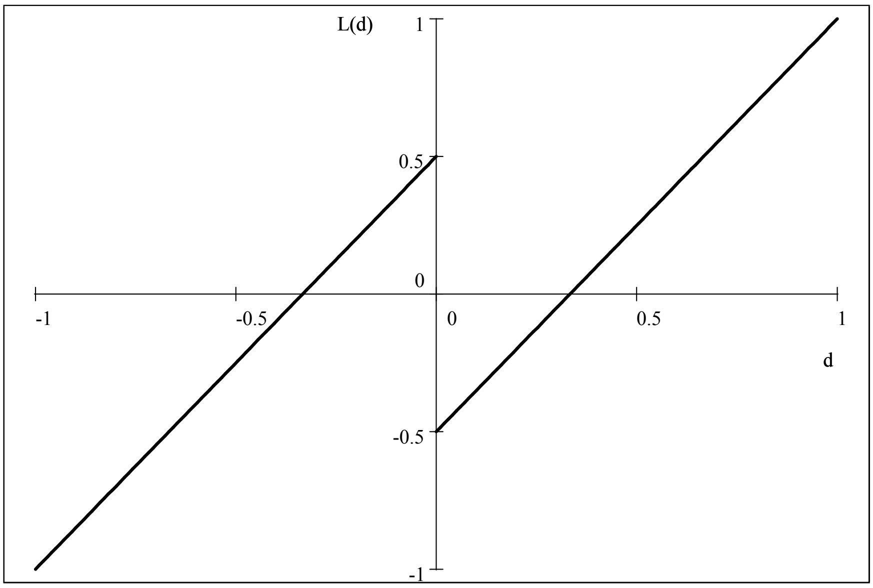
From our assumptions, L is continuous and non-decreasing on $[0, \overline{v} - \underline{v}]$, and

$$\begin{aligned} L(0) &< 0 \\ L(\overline{v} - \underline{v}) &= \overline{v} - \underline{v} > 0. \end{aligned}$$

Moreover, it is easy to check that L is an odd function[15] (whatever d belonging to $S_{\Delta} - \{0\}$, $L(d) = -L(-d)$).

Then from the above signs, there exists a t belonging to $[0, \overline{v} - \underline{v}]$, such that L is negative on $[\underline{v} - \overline{v}, -t] \cup [0, t]$, and positive on $[-t, 0] \cup [t, \overline{v} - \underline{v}]$.

[15] This comes from the symmetry of the joint distribution which induces that f_{Δ} is an even fonction and that whatever d belonging to S_{Δ}, $F_{\Delta}(d) = 1 - F_{\Delta}(-d)$.



The L function when buyers' valuations are independant and identically distributed according to an uniform distribution over $[0,1]$. Here, $t = 1/3$

Then an optimal increasing function P_1 will be such that, if $d < -t$, $P_1(d) = 0$, and if $d > t$, $P_1(d) = 1$ (maximizes their part of the integral without imposing any further constraint on the function P_1 on the remaining interval $[-t,t]$).

The most direct way to conclude is to note that whatever the increasing function P_1,

$$\begin{aligned}
\int_{[-t,t]} L(\delta) P_1(\delta) f_\Delta(\delta) d\delta &= \int_{[-t,0]} L(\delta) P_1(\delta) f_\Delta(\delta) d\delta + \int_{[0,t]} L(\delta) P_1(\delta) f_\Delta(\delta) d\delta \\
&= \int_{[0,t]} L(-u) P_1(-u) f_\Delta(-u) du + \int_{[0,t]} L(\delta) P_1(\delta) f_\Delta(\delta) d\delta \\
&= -\int_{[0,t]} L(u) P_1(-u) f_\Delta(u) du + \int_{[0,t]} L(\delta) P_1(\delta) f_\Delta(\delta) d\delta \\
&= \int_{[0,t]} L(\delta) (P_1(\delta) - P_1(-\delta)) f_\Delta(\delta) d\delta \\
&\leq 0, \text{ (as } P_1 \text{ is increasing and negative on } [0,t]) \\
&\leq \int_{[-t,t]} L(\delta) (1/2) f_\Delta(\delta) d\delta
\end{aligned}$$

So one[16] of the optimal P_1 is such $P_1(d) = 1/2$ on $[-t,t]$, $P_1(d) = 0$ on $[\underline{v} - \overline{v}, -t[$, and $P_1(d) = 1$ on

[16] Note that all the P_1 functions remaining constant on $[-t,t]$ are also optimal.

$]t, \overline{v} - \underline{v}]$.

- Step 6: Finds the corresponding payment rules and check that (IC) constraints hold.

For a direct revelation mechanisms, it must be true that:

$$\begin{aligned} T(d) &= dP_1(d) - w(d) \\ &= dP_1(d) - \int_0^d w'(\delta)\, d\delta - w(0) \\ &= dP_1(d) - \int_0^d P_1(\delta)\, d\delta + \underline{v}. \end{aligned}$$

So it is direct calculation to show that if $d < -t$ or $d > t$, then $T(d) = \underline{v} + t/2$, and that if $-t \leq d \leq t$,then $T(d) = \underline{v}$.

For this payment and allocation rule, the cartel will choose the offer $(P_1 = 1, T = \underline{v} + t/2)$ when

$$\begin{cases} v_1 - (\underline{v} + t/2) \geq \frac{v_1+v_2}{2} - \underline{v} \\ v_1 - (\underline{v} + t/2) \geq v_2 - (\underline{v} + t/2) \end{cases},$$

which occurs whenever

$$v_1 - v_2 \geq t.$$

Checking the valuations for which the cartel chooses the other offers shows that the (IC) constraints hold for this mechanism, which completes the proof. ■

9 Appendix 2

In order to present the equilibrium strategies, it will be easier to consider that the auction mechanism is structured in the following equivalent way: First the buyers are asked whether or not they want to participate the auction with reserve price r. Then, if any buyer has agreed to participate the auction with reserve price r, this auction is run, otherwise the item is randomly allocated at price $\underline{v}$.

9.1 The English auction implementation

Suppose buyer $B1$'s strategy is to participate the auction with reserve price r, if and only if his valuation v_1 exceeds v^*, and to stay at the auction until either the other drops the auction or the price reaches v_1. We'll show that Buyer $B2$'s best response is to follow the same strategy.

If Buyer $B2$'s valuation is less than r, he can't get a positive surplus from participating the auction with reserve price r, and so he does not.

Suppose now $v_2 \geq r$.

If Buyer $B2$ doesn't participate the auction with the reserve price r, he gets the item if and only if buyer $B1$ doesn't participate either (which happens when $v_1 < v^*$) and he wins the lottery (with a probability 1/2 and for a price $\underline{v}$). Thus his expected surplus if he doesn't participate the auction with reserve price is

$$\frac{(v_2 - \underline{v})}{2} F_m(v^*)$$

If Buyer $B2$ participates the auction with the reserve price r,the classical analysis shows that his best strategy during this auction is to stay until the price reaches v_2 or buyer $B1$ drops the auction. Following this strategy, his expected surplus if he participates the auction with reserve price is

$$\begin{aligned}
(v_2 - r) F_m(v^*) + \int_{v^*}^{\max(v_2,v^*)} & (v_2 - t) f_m(t) dt \\
&= (v_2 - r) F_m(v^*) + [(v_2 - t) F_m(t)]_{v^*}^{\max(v_2,v^*)} + \int_{v^*}^{\max(v_2,v^*)} F_m(t) dt \\
&= \begin{cases} (v_2 - r) F(v^*) & \text{if} \quad v_2 < v^* \\ (v^* - r) F_m(v^*) + \int_{v^*}^{v_2} F_m(t) dt & \text{if} \quad v_2 \geq v^* \end{cases}.
\end{aligned}$$

If $v_2 < v^*$, buyer $B2$ thus prefers not to participate the auction with reserve price r (as $\frac{(v_2 - \underline{v})}{2} F_m(v^*) > (v_2 - r) F_m(v^*)$). If $v_2 = v^*$, he is indifferent between participating the auction with reserve price or not, with both giving him the same expected surplus $(r - \underline{v}) F_m (v^*)$. Finally, when $v_2 > v^*$, the derivative according to v_2 of the expected surplus when he doesn't participate $(\frac{F_m(v^*)}{2})$ is lower than the derivative according to v_2 of the expected surplus when he participates $(F_m(v_2))$. So participating the auction with the reserve price only when $v_2 \geq v^*$ is a best response of buyer 2 to buyer 1's strategy. Symmetry completes the proof.

9.2 The second-price sealed bid implementation

Suppose buyer $B1$'s strategy is to participate the auction with the reserve price r, only if his valuation $v_1 \geq v^*$, and then to bid his true valuation v_1. We'll show that Buyer $B2$'s best response is to follow the same strategy.

The proof is much similar to the English auction implementation case. The key point is to note that if buyer $B2$ participates the auction with reserve price r, his best action is then to bid his true valuation v_2. So buyer $B2$'s expected surplus, according to whether or not he participates the auction with reserve price r is the same as for the English auction implementation: Hence, Buyer $B2$ participates the auction with the reserve price r only when $v_2 \geq v^*$.

9.3 The first-price sealed bid implementation

Here, we just give the idea of the proof.

Suppose buyer $B1$'s strategy is to participate the auction with the reserve price r, only if his valuation $v_1 \geq v^*$, and then to place the bid

$$b(v_1) = v_1 - \int_{\underline{v}}^{v_1} \frac{F_m(t)}{F_m(v_1)} dt - \frac{1}{F_m(v_1)}((v^* - r)F_m(v^*) - \int_{\underline{v}}^{v^*} F_m(t)dt)$$

(This function b is characterized as being the solution of the differential equation $F_m(v) = \frac{1}{b'(v)}(v - b(v))f_m(v)$ for which $b(v^*) = r$. It is easy to check that this function is non-decreasing).

If Buyer $B2$'s valuation is less than r, he can't get a positive surplus from participating the auction with reserve price r, so he prefers not to: Thus he benefits from the lottery in case $v_1 < v^*$, which gives him a positive surplus equal to

$$\frac{(v_2 - \underline{v})}{2} F\left(v^*\right)$$

Suppose now $v_2 \geq r$.

If buyer $B2$ participates the auction with reserve price r and places a bid bo (greater than r). Then his expected surplus is

$$(v_2 - bo)\, F_m(b^{-1}(bo))$$

Deriving this with respect to bo gives

$$-F_m(b^{-1}(bo)) + \frac{1}{b'(b^{-1}(bo))}\,(v_2 - bo)\, f_m(b^{-1}(bo))$$

which equals 0 for $bo = b(v_2)$ (from $b^{-1}(b(v_2)) = v_2$ and the characterization of the function b).

If we admit that this first order condition is enough to characterize buyer $B2$'s optimal bid[17], this proves that if buyer $B2$ participates the auction with reserve price r, his best bidding strategy is $b(v)$ if $b(v_2) \geq r$, that is if $v_2 \geq v^*$.

If $v_2 < v^*$, his best bidding strategy if he decides to participate is to bid r , which gives him an expected surplus equal to $(v_2 - r)\, F_m\left(v^*\right)$. Thus buyer $B2$ prefers not to participate when $v_2 < v^*$.

When $v_2 \geq v^*$, participating and bidding $b(v_2)$ gives him an expected surplus equal to

$$\begin{aligned}
(v_2 - b(v_2))\, F_m(v_2) &= \left(\int_{\underline{v}}^{v_2} \frac{F_m(t)}{F_m(v_2)} dt + \frac{1}{F_m(v_2)}((v^* - r)F_m(v^*) - \int_{\underline{v}}^{v^*} F_m(t)dt)\right)F_m(v_2) \\
&= \int_{\underline{v}}^{v_2} F_m(t)dt + (v^* - r)\, F_m\left(v^*\right) - \int_{\underline{v}}^{v^*} F_m(t)dt \\
&= (v^* - r)\, F_m\left(v^*\right) + \int_{v^*}^{v_2} F_m(t)dt
\end{aligned}$$

Thus when $v_2 = v^*$, he is indifferent between participating or not.

Since the derivative according to v_2 of the expected surplus when he participates $(F_m(v_2))$ is higher than the derivative according to v_2 of the expected surplus when he doesn't $(\frac{F_m(v^*)}{2})$, participating the auction with the reserve price only when $v_2 \geq v^*$ is a best response of buyer 2 to buyer 1's strategy.

[17] This is a bit long and can be shown using Milgrom and Weber's methodology [16].

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