Optimal auctions when a seller is bound to sell to collusive bidders

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OPTIMAL AUCTIONS WHEN A SELLER IS BOUND TO SELL TO COLLUSIVE BIDDERS*

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I consider optimal auctions for a seller who is bound to sell a single item to one of two potential buyers, organized in a 'well-coordinated' cartel. I show that, even though the seller cannot deter collusion, he can optimally accommodate it by employing a simple mechanism which imposes an inefficient allocation on the bidders unless they pay a sufficiently high amount to avoid it.

I. INTRODUCTION

MANY AUTHORS HAVE REPORTED EVIDENCE of collusion in auctions.[1] Collusion is clearly a major concern for the seller, whether his aim is revenue or efficiency maximization. Indeed, when the cartel is unable to use monetary transfers between its members, collusion can take the form of choosing the winner through randomization or by rotation among the cartel members, thus harming welfare.

Choosing an appropriate auction format may enable a seller to deter potential collusion. For instance, if there are few subsequent interactions among the buyers following the auction, and if monetary transfers between them are not possible, a sealed-bid auction can be effective against collusion. In this case, even if buyers reach an agreement prior to the auction, some will

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have incentives to deviate once the auction starts, and the other members of the cartel will not be able to react to this deviation). Moreover, some recent papers show that the seller can use information asymmetries between potentially colluding bidders in order to combat collusion.[2]

When certain types of item are repeatedly sold by different sellers but only a few bidders are interested in these items, there may be little information asymmetry between the bidders. One might take the example of concession contracts for services awarded by local public authorities, as in France, where only three major firms hold most of the concessions for water services.[3]

In these cases, it might seem that there is little the seller can do to increase its revenue apart from accommodating collusion, and setting high reserve prices to limit its adverse effects. However, this is clearly problematic when local public authorities or a government have to allocate concession contracts for valuable services that they could only provide themselves at a very high cost, or when the seller wants to obtain something it values highly (defense procurement, for example). Often, the seller is obliged to sell the item, even if at a very low price, so that setting a reserve price would not appear credible. Then, it might seem that the seller has no way of increasing its revenue above the maximum of the buyers' lowest valuations. In this paper, I show that this is not the case.

I identify the optimal auctions when the seller of a single item is bound to sell this item on the equilibrium path and is faced with a cartel that efficiently manages to coordinate its members' bids and participation decisions. I show that, if it can prevent resale of the item, the seller can maximize its revenue by running an auction which prescribes an inefficient allocation whenever the bidders offer a low payment.

To illustrate, consider two potential buyers $B1$ and $B2$, organized as an efficient cartel, who are potentially interested in an item which is to be sold at auction. The values of the item to the buyers are private and identically distributed, with a lowest possible valuation equal to $\underline{v}$. To increase its revenue above $\underline{v}$, the seller can run the following first-price sealed-bid auction. If at least one of the buyers places a bid above a certain reserve price r, then the item is sold to the buyer who makes the highest bid. Otherwise, the item is either sold at price $\underline{v}$ to the winner of a lottery, or not sold at all if one of the buyers has refused to participate in the auction. If the cartel is efficient, both buyers will participate in the auction and no buyer will ever place a bid higher than the reserve price r. Hence, the cartel must choose between accepting a random allocation of the item at a low price $\underline{v}$ or 'buying' a more efficient allocation (i.e. giving the item to the buyer with the highest valuation) at a higher price r. If the difference between the buyers' valuations is large compared to $r - \underline{v}$, the cartel will accept to pay a higher price r, to ensure that the item goes to the buyer with highest valuation.

After briefly reviewing the literature in Section II, the model is presented in Section III. In Section IV, I derive the optimal auction mechanisms for symmetric and asymmetric valuation distributions, and show that an optimal auction entails a 'fair amount' of competition even if the bidders do not collude. For symmetric valuation distributions, I suggest using a standard auction with reserve price, and randomly allocating the item at a price equal to the minimum possible valuation in case no bid exceeds the reserve price. Section V extends this implementation to an n-buyer auction and concludes.

II. LITERATURE REVIEW

Graham and Marshall [1987] and McAfee and McMillan [1992] were among the first to study collusion in auctions. They analyze the optimal collusion mechanism, and show that a cartel usually manages to efficiently coordinate its members' bids and participation decisions, for one-shot second-price auctions and one-shot first-price auctions with reserve price respectively.

Following this work, Aoyagi [2003], Blume and Heidhues [2008], and Skrzypacz and Hopenhayn [2004] study collusion in repeated auctions. These papers show that, if the seller must choose the auction format among relatively 'standard' subsets of auction mechanisms, then, while cartels may sometimes fail to efficiently coordinate the bids, they usually manage to get quite close to efficiency, if the bidders are sufficiently patient.

These papers may give the impression that little can be done to fight collusion in auctions, apart from accommodating it with reserve prices. However, a new branch of the literature shows that this is not the case: Following two path-breaking papers by Laffont and Martimort [1997, 2000], some articles look for optimal auctions without restricting attention to a limited subset of mechanisms, and focus on how the seller can exploit the colluders' asymmetric information to prevent them from coordinating their behaviour. Interestingly, Che and Kim [2006A, 2006B], Pavlov [2006] and Quesada [2005] show that the seller can achieve the same expected revenue as in the case of non-collusion in a wide range of circumstances.[4] As highlighted by Che and Kim [2006B], these positive results rely crucially on not selling the item for some combinations of the buyers' types.

Consequently, the literature seems to suggest that a seller facing potential collusion has two main ways to improve revenue: (1) by using information asymmetry between the bidders to 'break' the cartel, and (2) by threatening not to sell the item if all the buyers have a low valuation. I study collusion in one of the worst possible case scenario for the seller: That is, when there is no information asymmetry between the buyers, and when the seller has to ensure that the item is always sold on the equilibrium path (so that excluding some of the buyers types from buying the item is impossible).[5]

III. THE MODEL

III(i). *Primitives*

A risk-neutral seller has to sell an item through an auction mechanism that it can design. There are two potential buyers $B1$ and $B2$. Let v_1 and v_2 be buyer $B1$'s and buyer $B2$'s private valuations of the item, drawn from a given continuous joint density function, with support $S = [\underline{v}_1, \overline{v}_1] \times [\underline{v}_2, \overline{v}_2]$.[6] Without loss of generality, I will consider that $\underline{v}_2 \leq \underline{v}_1$. I also assume $\underline{v}_1 < \overline{v}_2$, so that buyer $B2$ may potentially value the item higher than buyer $B1$. Buyers are risk-neutral.

Let $d = v_1 - v_2$ be the difference between the two buyers' valuations, f_Δ and F_Δ be its density and distribution function on $S_\Delta = [\underline{v}_1 - \overline{v}_2, \overline{v}_1 - \underline{v}_2]$. I also assume that the functions J_1 and J_2, defined by $J_1(d) = d + \frac{F_\Delta(d)}{f_\Delta(d)}$ on $[\underline{v}_1 - \overline{v}_2, \underline{v}_1 - \underline{v}_2]$, and $J_2(d) = d - \frac{1-F_\Delta(d)}{f_\Delta(d)}$ on $[\underline{v}_1 - \underline{v}_2, \overline{v}_1 - \underline{v}_2]$, are non-decreasing. This is a classic assumption in auction theory and is verified using the most common distribution functions F_Δ.[7]

III(ii). *Model of collusion and time line*

I assume that, once the seller has proposed an auction mechanism, the two buyers can choose to form a cartel to coordinate their bids and participation decisions. I also assume that each buyer learns about the other buyer's valuation and that, if a cartel is formed, buyers bargain efficiently over their bids and participation decisions, and arrange a monetary transfer between them. For example, the bids, participation decisions and monetary transfers may be thought of as being obtained as the solution to a Nash-Bargaining problem with exogenous bargaining power (which does not need to be specified) and outside options equal to the buyers' non-cooperative payoffs. Thus, if a cartel is formed, the buyers' bids and participation decisions will maximize the sum of the buyers' payoffs. I further assume that the seller cannot observe whether or not a cartel is formed.

The time line of the game is as follows:

- At date 0, both buyers learn of the two valuations v_1 and v_2.[8]

- At date 1, the seller proposes an auction mechanism.

- At date 2, the buyers simultaneously choose whether or not to form a cartel. If both decide to form a cartel, the cartel is active, otherwise it is inactive.

- At date 3, each buyer participates, or not, in the auction and bids if he participates.

- At date 4, the outcome of the auction mechanism arises (along with monetary transfers between the buyers if they form a cartel).

Whatever the auction mechanism is and whatever the bidders' valuations are, it is clear that both buyers will be willing to form a cartel at date 2. Indeed if a cartel is formed, each buyer eventually gets a payoff above the payoff he would get if both buyers competed. So, I focus on the equilibrium where both buyers always decide to form a cartel.

Consequently, this game has the same outcome as a new simplified game between the seller and a third party, the cartel, which coordinates both players' actions (participation and bids) in order to maximize the sum of their payoffs. The time line of this simplified game is:

- At date 0, the cartel learns of the two valuations v_1 and v_2.

- At date 1, the seller proposes an auction mechanism.

- At date 2, the cartel decides which buyers participate and announces the bids of the participants.

- At date 3, the outcome of the auction mechanism is given.

In what follows, I focus on this simplified game. As usual, all the assumptions are common knowledge.

III(iii). *Set of feasible mechanisms*

I assume that the seller can regulate resale, so that the mechanism can state who ends up consuming the item. Hence, the cartel realizes value v_i, when buyer Bi wins the auction.

Moreover, I assume that the seller needs to sell the item with probability 1 on the equilibrium path. For some potential actions, the mechanism may state that the item is not allocated, but it must give the correct incentives for the cartel to always choose actions such that the item is allocated, whatever the buyers' valuations.

It is possible for the cartel to have only one bidder participate in the auction, although this possibility can be ignored without any loss of generality. The reason is that, if an optimal auction involves such a possibility, one can construct another payoff-equivalent (and thus optimal) auction which always induces both parties to participate.

To understand this, imagine an optimal auction mechanism such that for some (v_1', v_2') belonging to S, the cartel has only one of the buyers, say buyer $B2$, participating in the auction. In this case buyer $B2$ gets the item with probability 1 upon payment of T_2'. Then, we can modify the mechanism by offering the possibility for the cartel to have buyer $B2$ winning the auction with payment T_2' when both buyers participate, and by stating that the item is not sold if buyer $B1$ does not participate. This new mechanism is also optimal since its outcome is the same as the original one.

Consequently, we can restrict our attention to mechanisms whereby the item is not sold when only one buyer participates, so that the cartel always has both buyers participating.[9]

By the revelation principle (Myerson [1981]), there is no loss in restricting attention to direct revelation mechanisms. Let $P_i(\widehat{v}_1, \widehat{v}_2)$ denote the probability that buyer Bi gets the item if the cartel announces a valuation $\widehat{v}_1$ for buyer $B1$ and a valuation $\widehat{v}_2$ for buyer $B2$. Let $T(\widehat{v}_1, \widehat{v}_2)$ denote the corresponding expected payment (including what both buyer $B1$ and buyer $B2$ have to pay).

We look for the mechanism (P_1, P_2, T) which generates the highest expected revenue for the seller among those that satisfy the three constraints:

(IC) constraints:

$$\begin{aligned} \forall\left((v_1, v_2), (\widehat{v}_1, \widehat{v}_2)\right) &\in S^2, \\ v_1 P_1(v_1, v_2) + v_2 P_2(v_1, v_2) - T(v_1, v_2) &\geq v_1 P_1(\widehat{v}_1, \widehat{v}_2) + v_2 P_2(\widehat{v}_1, \widehat{v}_2) - T(\widehat{v}_1, \widehat{v}_2) \end{aligned}$$

(IR) constraints:

$$\forall (v_1, v_2) \in S, \quad v_1 P_1(v_1, v_2) + v_2 P_2(v_1, v_2) - T(v_1, v_2) \geq 0$$

(Allocation) constraints:

$$\forall (v_1, v_2) \in S, \quad P_1(v_1, v_2) \geq 0, \; P_2(v_1, v_2) \geq 0, \text{ and } P_1(v_1, v_2) + P_2(v_1, v_2) = 1.$$

The (IC) constraint means that the cartel has no better strategy than announcing the buyers' true valuations. The (IR) constraint means that whatever its valuations, the cartel will be willing to participate in the auction. The (Allocation) constraint is necessary to ensure that the item will be sold whatever the buyers' valuations.

IV. ANALYSIS

First, note that the (IC) constraints imply that if the (IR) constraint holds for the minimum valuations $(\underline{v}_1, \underline{v}_2)$, then it holds for any v_1 and v_2.[10] Thus, the (IR) constraint can be rewritten as

$$\underline{v}_1 P_1(\underline{v}_1, \underline{v}_2) + \underline{v}_2 P_2(\underline{v}_1, \underline{v}_2) - T(\underline{v}_1, \underline{v}_2) \geq 0.$$

For an optimal mechanism, this constraint must be binding at this point, since otherwise the seller could increase the payment T by a constant $\epsilon > 0$ for every type without upsetting the (IC) and (allocation) constraints. So, we can restrict our attention to mechanisms for which

$$\underline{v}_1 P_1(\underline{v}_1, \underline{v}_2) + \underline{v}_2 P_2(\underline{v}_1, \underline{v}_2) - T(\underline{v}_1, \underline{v}_2) = 0. \tag{1}$$

Due to the (allocation) constraint, this two-dimensional problem can be transformed into a one-dimensional problem.

Lemma 1. For every direct revelation mechanism, there exists another direct revelation mechanism which yields at least as much revenue and such that P_1, P_2 and T are constant on any given line $v_1 - v_2 = d$.

Proof. Using the allocation constraint, (IC) implies that for any (v_1, v_2) and $(\widehat{v}_1, \widehat{v}_2)$ belonging to S,

$$\begin{aligned}(v_1 - v_2) P_1(v_1, v_2) - T(v_1, v_2) + v_2 \\ \geq (v_1 - v_2) P_1(\widehat{v}_1, \widehat{v}_2) - T(\widehat{v}_1, \widehat{v}_2) + v_2,\end{aligned}$$

or, similarly,

$$(v_1 - v_2)\, P_1 (v_1, v_2) - T (v_1, v_2) \geq (v_1 - v_2)\, P_1 (\widehat{v}_1, \widehat{v}_2) - T (\widehat{v}_1, \widehat{v}_2) .$$

This inequality must also hold if we switch v and $\widehat{v}$, which gives

$$(\widehat{v}_1 - \widehat{v}_2)\, P_1 (\widehat{v}_1, \widehat{v}_2) - T (\widehat{v}_1, \widehat{v}_2) \geq (\widehat{v}_1 - \widehat{v}_2)\, P_1 (v_1, v_2) - T (v_1, v_2) .$$

Combining these two inequalities shows that, if $v_1 - v_2 = \widehat{v}_1 - \widehat{v}_2$, then

$$\begin{aligned} (v_1 - v_2)\, P_1 (v_1, v_2) - T (v_1, v_2) &\geq \\ (v_1 - v_2)\, P_1 (\widehat{v}_1, \widehat{v}_2) & \quad -T (\widehat{v}_1, \widehat{v}_2) \\ &\geq (v_1 - v_2)\, P_1 (v_1, v_2) - T (v_1, v_2) , \end{aligned}$$

so that

$$(v_1 - v_2)\, P_1 (v_1, v_2) - T (v_1, v_2) = (v_1 - v_2)\, P_1 (\widehat{v}_1, \widehat{v}_2) - T (\widehat{v}_1, \widehat{v}_2) .$$

This shows that for a direct revelation mechanism, the cartel is indifferent between the offer that is designed for its type (v_1, v_2) and offers that are designed for other types $(\widehat{v}_1, \widehat{v}_2)$ on the same line $v_1 - v_2 = \widehat{v}_1 - \widehat{v}_2$.

Let $T(d)$ be the least upper bound of the $T(v_1, v_2)$ on the line $v_1 - v_2 = d$:

$$T(d) = \sup \{ T (v_1, v_2) \,| v_1 - v_2 = d \} .$$

Let $P_1(d)$ be such that for any (v_1, v_2) on the line $v_1 - v_2 = d$, we have

$$(v_1 - v_2)\, P_1 (v_1, v_2) - T (v_1, v_2) = (v_1 - v_2)\, P_1 (d) - T (d) .$$

Changing all offers designed for types on the line $v_1 - v_2 = d$, from $P_1(v_1, v_2)$ to $P_1(d)$ and from $T(v_1, v_2)$ to $T(d)$ increases the expected revenue generated by the mechanism and does not violate any of our three constraints.[11] ■

Consequently, we can restrict our attention to direct revelation mechanisms that have the same allocation and payment rules $(P_1(d), T(d))$ everywhere inside any given line $v_1 - v_2 = d$. From now on, P_1, P_2 and T will be considered to be functions of d.[12]

In this context, as $P_1(.) + P_2(.) = 1$, our three constraints can be written with respect to d:

(IC)' constraints:

$$\forall \left(d, \widehat{d} \right) \in S_{\Delta}^2, \quad dP_1 (d) - T (d) \geq dP_1 \left(\widehat{d} \right) - T \left(\widehat{d} \right) .$$

(IR)' constraints:

$$(\underline{v}_1 - \underline{v}_2)P_1\left(\underline{v}_1 - \underline{v}_2\right) - T\left(\underline{v}_1 - \underline{v}_2\right) = -\underline{v}_2.$$

(Allocation)' constraints:

$$\forall d \in S_\Delta, \quad P_1(d) \in [0,1] \text{ and } P_2(d) = 1 - P_1(d).$$

Now that we have a one-dimensional problem, Myerson's methodology ([1981]) can be applied. Let $L : S_\Delta \longrightarrow R$ be the function defined by:

$$L \begin{cases} d + \frac{F_\Delta(d)}{f_\Delta(d)} & \text{if } d < \underline{v}_1 - \underline{v}_2, \\ d - \frac{1-F_\Delta(d)}{f_\Delta(d)} & \text{if } d \geq \underline{v}_1 - \underline{v}_2. \end{cases}$$

Lemma 2. Every mechanism satisfying (IC)', (IR)' and (Allocation)' is such that P_1 is non-decreasing in d. The seller's expected revenue is

$$E(T) = \int_{S_\Delta} L(\delta)\, P_1(\delta)\, f_\Delta(\delta)\, d\delta + \underline{v}_2. \tag{2}$$

Proof. Whatever d and $\widehat{d}$ belonging to S_Δ, (IC)' constraints imply that

$$dP_1(d) - T(d) \geq dP_1\left(\widehat{d}\right) - T\left(\widehat{d}\right) \text{ and } \widehat{d}P_1\left(\widehat{d}\right) - T\left(\widehat{d}\right) \geq \widehat{d}P_1(d) - T(d).$$

Thus, we have

$$d(P_1\left(\widehat{d}\right) - P_1(d)) \leq T\left(\widehat{d}\right) - T(d) \leq \widehat{d}(P_1\left(\widehat{d}\right) - P_1(d)), \tag{3}$$

which implies that

$$0 \leq \left(\widehat{d} - d\right)(P_1\left(\widehat{d}\right) - P_1(d)).$$

So, the function P_1 is non-decreasing in d. Let w be the function defined by $w(d) \equiv dP_1(d) - T(d)$.

From inequality 3, we have

$$(d - \widehat{d})P_1\left(\widehat{d}\right) \leq w(d) - w(\widehat{d}) \leq (d - \widehat{d})P_1(d).$$

This proves that w is continuous and differentiable at every d for which P_1 is continuous (which happens everywhere except on a countable set of points, as P_1 is non-decreasing). Moreover, for these points we have $w'(d) = P_1(d)$. Since the payment T is a function of d, the seller's expected revenue can be expressed by integrating over the values of d, so that

$$\begin{aligned} E(T) &= \int_{S_\Delta} T(\delta) f_\Delta(\delta)\, d\delta \\ &= \int_{S_\Delta} \delta P_1(\delta) f_\Delta(\delta)\, d\delta - \int_{S_\Delta} w(\delta) f_\Delta(\delta)\, d\delta. \end{aligned}$$

As the function $\omega(.) F_\Delta(.)$ is continuous, differentiable everywhere except on a countable set of points, it is a generalized primitive of the function $\omega(.) f_\Delta(.) + P_1(.) F_\Delta(.)$ and we can integrate the second integral by parts. So,

$$\begin{aligned} E(T) &= \int_{S_\Delta} \delta P_1(\delta) f_\Delta(\delta)\, d\delta - [w(\delta) F_\Delta(\delta)]_{\underline{v}_1-\overline{v}_2}^{\overline{v}_1-\underline{v}_2} + \int_{S_\Delta} P_1(\delta) F_\Delta(\delta)\, d\delta \\ &= \int_{S_\Delta} (\delta f_\Delta(\delta) + F_\Delta(\delta)) P_1(\delta)\, d\delta - w(\overline{v}_1 - \underline{v}_2) \\ &= \int_{S_\Delta} (\delta f_\Delta(\delta) + F_\Delta(\delta)) P_1(\delta)\, d\delta - (w(\overline{v}_1 - \underline{v}_2) - w(\underline{v}_1 - \underline{v}_2)) - w(\underline{v}_1 - \underline{v}_2) \\ &= \int_{S_\Delta} (\delta f_\Delta(\delta) + F_\Delta(\delta)) P_1(\delta)\, d\delta - \int_{\underline{v}_1-\underline{v}_2}^{\overline{v}_1-\underline{v}_2} P_1(\delta)\, d\delta - w(\underline{v}_1 - \underline{v}_2) \\ &= \int_{S_\Delta} L(\delta) P_1(\delta) f_\Delta(\delta)\, d\delta + \underline{v}_2, \end{aligned}$$

as $w(\underline{v}_1 - \underline{v}_2) = -\underline{v}_2$ from (IR)'. ■

Hence, our problem amounts to finding the function P_1 which maximizes $\int_{S_\Delta} L(\delta) P_1(\delta) f_\Delta(\delta)\, d\delta$, among the non-decreasing functions taking their values in $[0,1]$. We are now in a position to identify the optimal mechanism.

Proposition 1. Let s and t be such that

$$\begin{aligned} s &:= \inf \{d \in [\underline{v}_1 - \overline{v}_2, \underline{v}_1 - \underline{v}_2] \mid L(d) \geq 0\}, \\ t &:= \inf \{d \in [\underline{v}_1 - \underline{v}_2, \overline{v}_1 - \underline{v}_2] \mid L(d) \geq 0\}. \end{aligned}$$

If $\int_s^t L(\delta) f_\Delta(\delta)\, d\delta \geq 0$, then an optimal auction mechanism either allocates the item to buyer $B1$ at price $\underline{v}_1$, or to buyer $B2$ at a higher price $\underline{v}_1 - s$. If $\int_s^t L(\delta) f_\Delta(\delta)\, d\delta \leq 0$, then an optimal auction mechanism either allocates the item to buyer $B2$ at price $\underline{v}_2$, or to buyer $B1$ at a higher price $\underline{v}_2 + t$.

Proposition 2. If the joint distribution of the valuations is symmetric, there exists a price r , where $\underline{v} < r < \underline{v} + \frac{\overline{v}-\underline{v}}{2}$, such that an optimal auction mechanism is to propose two possible outcomes:

1. either randomly allocate the item to one of the two buyers (with equal probability) at price $\underline{v}$,

2. or let the cartel choose who gets the item at a higher price r.

   This price r is such that $r = \underline{v} + \frac{x}{2}$, where x is the unique solution to

   $$x = \frac{1 - F_{\Delta}(x)}{f_{\Delta}(x)}.$$

   This leads the item to be allocated randomly when the difference between the buyers' valuations is less than x, and to the buyer with the highest valuation otherwise.

Proof. To prove propositions 1 and 2, I first look for the non-decreasing function P_1 which maximizes the seller's revenue (2) without taking the (IC)' constraint into account. Then, I check that the solution satisfies this constraint.

From our assumptions, L is continuous and non-decreasing on $[\underline{v}_1 - \overline{v}_2, \underline{v}_1 - \underline{v}_2]$ and on $[\underline{v}_1 - \underline{v}_2, \overline{v}_1 - \underline{v}_2]$, and

$$\begin{aligned} L(\underline{v}_1 - \overline{v}_2) &< 0, \\ \lim_{t \to (\underline{v}_1 - \underline{v}_2)^-} L(t) &\geq \underline{v}_1 - \underline{v}_2 \geq 0, \\ L(\overline{v}_1 - \underline{v}_2) &> 0. \end{aligned}$$

Moreover, it is clear that $s := \inf \{d \in [\underline{v}_1 - \overline{v}_2, \underline{v}_1 - \underline{v}_2] \mid L(d) \geq 0\}$ is negative, while $t := \inf \{d \in [\underline{v}_1 - \underline{v}_2, \overline{v}_1 - \underline{v}_2] \mid L(d) \geq 0\}$ is positive. Thus, we have to maximize $\int_{S_\Delta} L(\delta) f_\Delta(\delta) P_1(\delta) d\delta$, with $L(\delta) f_\Delta(\delta)$ being negative on $[\underline{v}_1 - \overline{v}_2, s)$, non-negative on $(s, \underline{v}_1 - \underline{v}_2)$, negative on $(\underline{v}_1 - \underline{v}_2, t)$ and non-negative on $(t, \overline{v}_1 - \underline{v}_2]$.

First, it is clear that an optimal function P_1 will be such that, if $d < s$, then $P_1(d) = 0$, and if $d > t$, then $P_1(d) = 1$ (these values maximize the parts of the integral over $[\underline{v}_1 - \overline{v}_2, s)$ and $(t, \overline{v}_1 - \underline{v}_2]$, without imposing any further constraint on the function P_1 on the remaining interval $[s, t]$). Then,

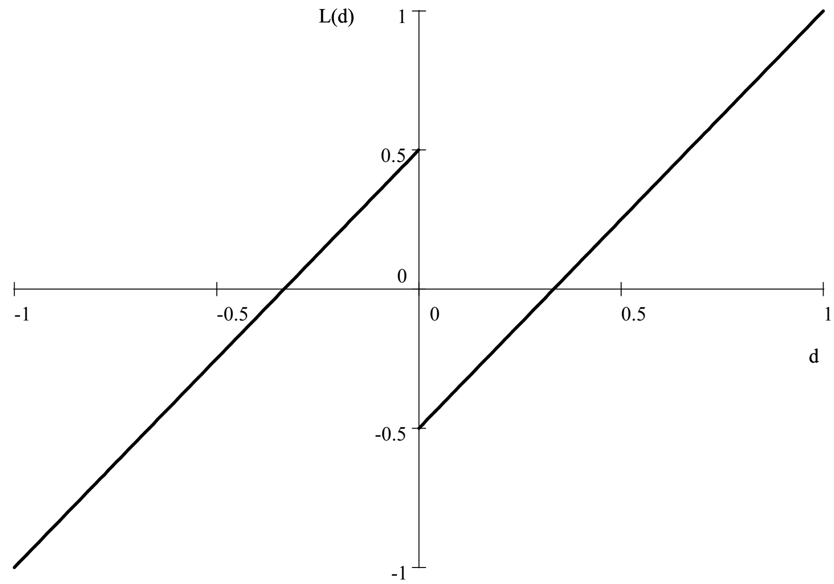

Figure 1: The L function when buyers' valuations are independent and identically distributed according to an uniform distribution over $[0,1]$. Here, $t = 1/3$.

one of the optimal functions P_1 is constant on (s,t). Indeed if there was a value $x \in (s, \underline{v}_1 - \underline{v}_2)$ such that $P_1(x) < P_1(\underline{v}_1 - \underline{v}_2)$, one could improve the seller's revenue by taking $P_1 = P_1(\underline{v}_1 - \underline{v}_2)$ on $(s, \underline{v}_1 - \underline{v}_2)$, since in this range $L(.)\, f_\Delta(.)$ is non-negative. Following the same logic, if there was a $x \in (\underline{v}_1 - \underline{v}_2, t)$ such that $P_1(x) > P_1(\underline{v}_1 - \underline{v}_2)$, one could improve the seller's revenue by taking $P_1 = P_1(\underline{v}_1 - \underline{v}_2)$ on $(\underline{v}_1 - \underline{v}_2, t)$, since on this interval $L(.)\, f_\Delta(.)$ is negative.

Denoting by a the value of P_1 on (s,t), we can write the seller's expected payment as

$$\int_{S_\Delta} L(\delta)\, f_\Delta(\delta)\, P_1(\delta)\, d\delta + \underline{v}_2 = a \int_s^t L(\delta)\, f_\Delta(\delta)\, d\delta + \int_t^{\overline{v}_1 - \underline{v}_2} L(\delta)\, f_\Delta(\delta)\, d\delta + \underline{v}_2.$$

If $\int_s^t L(\delta)\, f_\Delta(\delta)\, d\delta \geq 0$, then an optimal mechanism is such that $P_1 = 0$ on $[\underline{v}_1 - \overline{v}_2, s)$ and $P_1 = 1$ on $(s, \overline{v}_1 - \underline{v}_2]$. If, on the contrary, $\int_s^t L(\delta)\, f_\Delta(\delta)\, d\delta \leq 0$, then an optimal mechanism is such that $P_1 = 0$ on $[\underline{v}_1 - \overline{v}_2, t)$ and $P_1 = 1$ on $(t, \overline{v}_1 - \underline{v}_2]$.

If the joint distribution of the valuations is symmetric, it can easily be shown that $\int_s^t L(\delta)\, f_\Delta(\delta)\, d\delta = 0$ and $s = -t$. Another optimal mechanism is such that $P_1 = 0$ on $[\underline{v} - \overline{v}, -t)$, $P_1 = 1/2$ on $[-t, t]$ and $P_1 = 1$ on $(t, \overline{v} - \underline{v}]$.[13]

Using the definition of the function w, the corresponding payments can then be computed:

$$\begin{aligned}
T(d) &= dP_1(d) - w(d) \\
&= dP_1(d) - \int_{\underline{v}_1 - \underline{v}_2}^{d} w'(\delta)\, d\delta - w(\underline{v}_1 - \underline{v}_2) \\
&= dP_1(d) - \int_{\underline{v}_1 - \underline{v}_2}^{d} P_1(\delta)\, d\delta + \underline{v}_2.
\end{aligned}$$

Direct calculation over the payments makes it possible to check that the (IC) constraints hold for these mechanisms and completes the proof. ■

In case the bidders are asymmetric, an optimal mechanism allocates the item to one of the buyers, say buyer Bi, for a price equal to its reservation valuation $\underline{v}_i$, and offers the cartel the chance to 'buy' an efficient allocation by granting the item to buyer Bj if buyer Bj values the item highly.[14] In the case of a symmetric valuations distribution, another optimal mechanism would be to randomize the winner of the auction for a price equal to the reservation valuation $\underline{v}$, except when the cartel is willing to pay an extra price to ensure that the item goes to the buyer who values it the most. In both cases, the idea is to impose an inefficient allocation on the bidders unless they pay a sufficiently high amount to avoid it.[15]

This result is quite intuitive, and might seem obvious, yet it is not. A priori, the optimal auction mechanism might have consisted of a complex list of proposed random allocations with different corresponding payments. Surprisingly, whatever the valuations' distribution, offering the above two possible outcomes is enough to maximize the seller's revenue.

Note that to implement this mechanism, the seller does not have to address the cartel as an entity. A simple implementation of the symmetric mechanism is to run a standard sealed-bid first-price or second-price auction with reserve price r ($\underline{v} < r < \underline{v} + \frac{\overline{v}-\underline{v}}{2}$) and, in case both buyers place bids below r, to have a lottery determine the winner at a price $\underline{v}$. Furthermore, the seller threatens not to sell the item if one of the buyers does not participate in the auction mechanism, unless the other buyer places a bid above r. With this auction, a cartel has to choose between outcomes where the winner is randomly drawn and outcomes where the winner is determined by the highest bid. The easiest way to ensure that a particular buyer wins the auction is to have it bid r while the other places a bid below r. Hence, the cartel has to choose between randomizing the winner for a price equal to $\underline{v}$ or choosing the winner for a price equal to r. Facing the same trade-off as with my optimal mechanism, a cartel thus has an interest in choosing bids which eventually yield the same outcome as in my mechanism.

Interestingly, if buyers happen to compete instead of colluding, this mechanism may very well yield nearly as much revenue as the optimal 'competitive' auction. Indeed, it can be shown that if the buyers' valuations are uniformly distributed on $[\underline{v}, \underline{v} + l]$, the seller's expected revenue when two buyers compete is $\underline{v} + 26/27 \times l/3$ with my mechanism, while it would be $\underline{v} + l/3$ for the optimal 'competitive' auction (that is any of the four standard auctions without reserve price, which are optimal when there is competition and when the item always has to be sold). In case of collusion, the seller's expected revenue is respectively equal to $\underline{v} + 2/9 \times l/3$ with my mechanism and $\underline{v}$ with the optimal 'competitive' auction.

Note that for asymmetric valuation distributions, my optimal asymmetric mechanism could easily be implemented by following the same logic. A standard auction, say a sealed-bid first price auction, is held only if both buyers participate. If this auction yields revenues over a given reserve price r', then the item is allocated to the winner of the auction, otherwise buyer Bi receives the item for a payment equal to $\underline{v}_i$.[16]

V. CONCLUDING REMARKS

In this paper, we saw that when a seller is bound to sell an item to one of two collusive bidders, the best it can do to improve its revenues is either to have the cartel choose a lottery, or pay for excluding one buyer.

Thus, a natural extension of this mechanism in the case of n potential buyers would be to plan a lottery between all the buyers, and to offer the cartel the possibility of paying to exclude one or more buyers from this lottery before it is run. It would then consist of n 'reserve'-prices: $r_n = \underline{v} \leq r_{n-1} \leq r_{n-2} \leq .... \leq r_2 < r_1$, where r_i is the price to be paid for the item if the cartel decides that only i buyers participate in the lottery.

In order to implement this mechanism, so as to benefit from competition if buyers do not collude, a standard auction can be slightly modified. First, a classic auction is run. Then, we look for the lowest k such that more than k bids are above the 'reserve'-price r_k. If no k satisfies this condition, the item is not sold.[17] Otherwise, a lottery between the k buyers who have submitted the highest bids determines who gets the item for the price that would have been paid in the standard auction by the buyer with the k^{th} highest bid (given reserve price r_k).

In fact, this could also be implemented sequentially. A first standard auction is run (say, a first price sealed bid auction) with reserve price r_1. If no buyer participates in this auction, then a second auction is run, with a lower reserve price r_2. If two or more buyers participate in this auction the item is sold via a lottery to one of the two highest bidders for a price equal to the second highest bid. Otherwise, a third auction is run, with a lower reserve price r_3. If three or more buyers participate in this auction, the item is sold via a lottery to one of the three highest bidders for a price equal to the third highest bid, and so on.

All of the above mechanisms rely on the threat of not selling the item in case some of the buyers refuse to participate. However, since we assumed that the item always had to be sold on the equilibrium path, these mechanisms are designed so that any type of cartel finds an advantage in having all of the buyers participating in the auction. In practice, this might cause a problem since buyers know that the seller will be tempted to run a second auction if the item is not sold after the first auction. One potential way to solve this problem if the buyers frequently interact in similar auctions, for example for concession contracts run by many different local public authorities, might be for the government of the country to require that buyers first have a license before being allowed to participate in such auctions. This license might then be granted conditionally upon the buyers committing to participate in every concession contract auction run in the country as long as the auction's reserve price does not exceed a certain amount.

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FOOTNOTES

1) See Baldwin, Marshall, and Richard [1997], Comanor and Schankerman [1976], Cook [1963], Graham and Marshall [1987], Hay and Kelley [1974], Marshall and Meurer [2004], Pesendorfer [2000], Porter and Zona [1999], Phillips, Menkhaus and Coatney [2003], Mund [1960], and Stigler [1964].

2) Che and Kim [2006A, 2006B], Dequiedt [2007], Laffont and Martimort [1997, 2000], Pavlov [2006], Quesada [2005].

3) Chong et al. [2006] show that the final price paid by consumers for these services tends to be higher when concessions are granted than when the service is directly managed by the local public authority.

4) In particular, these authors obtain positive results whether collusion on participation is possible as in Pavlov [2006] and Che and Kim [2006B], or not, as in Che and Kim [2006A]. Dequiedt [2007] finds a negative result, but as noted by Che and Kim [2006B], this difference seems to come from the particular setting of his model, where there are only two possible types of buyer.

5) To our knowledge, Mookherjee and Tsumagari [2004] is the only paper which deals with such a case, albeit in a different context.

6) Note that the valuations are not necessarily independent.

7) One could wonder whether or not this is still verified for distribution functions F_{Δ} deriving from most standard valuations distributions. However, we could relax this assumption, by assuming that J_i equals 0 only for a finite number of values, but this would make the description of our results more complex without providing additional insight into our problem.

8) This assumption may not seem realistic, yet in some markets, there are only a few buyers who frequently interact in auctions for items which are sold by different sellers but whose characteristics are very similar. In such cases, each buyer may eventually come to have very good information about his competitors' valuations.

9) In this model, the seller can commit not to sell the item if only one buyer participates, while it is bound to sell the item on the equilibrium path. As each buyer can choose whether or not to participate in the auction, this possibility is required to have the buyers pay more than their lowest possible valuation. Otherwise, a cartel could manipulate any mechanism by sending only one bidder and having him declare that his valuation is the lowest possible one. Hence, this assumption is only needed to ensure that both buyers will participate in the auction. This could be justified in

situations where selling the item is important enough to discourage the seller from using reserve prices to raise its revenue, but not important enough to make up for the consequences of a reputation loss if it should commit not to sell the item and renege. Finally, an alternative assumption would be to consider that the seller can force each of the buyers to participate in the auction, provided that this auction ensures a positive surplus for any type of buyer (this would amount to forcing each buyer to accept buying the item for its lowest possible valuation if required to do so by the seller).

10) To understand this, take $\widehat{v}_i = \underline{v}_i$ in the (IC) constraint. This result comes from P_i being non-negative and v_i being higher than $\underline{v}_i$.

11) Note that this change does not impact the payoff of a cartel whose type $(\widehat{v}_1, \widehat{v}_2)$ lies on the line $v_1 - v_2 = d$, if it truthfully reports its type. Thus, such a cartel has no more incentive to misreport his type after the change than he had before. Further, a cartel whose type $(\widehat{v}_1, \widehat{v}_2)$ does not lie on the line $v_1 - v_2 = d$, still has no incentive to misreport his type after the change. Indeed, $(P_1(d), T(d))$ can be seen as the limit of a sequence $(P_1(v_{1,n}, v_{2,n}), T(v_{1,n}, v_{2,n}))$, with $(v_{1,n}, v_{2,n})$ belonging to the line $v_1 - v_2 = d$. 'Former' (IC) constraints ensured that $(\widehat{v}_1 - \widehat{v}_2) P_1(\widehat{v}_1, \widehat{v}_2) - T(\widehat{v}_1, \widehat{v}_2) \geq (\widehat{v}_1 - \widehat{v}_2) P_1(v_{1,n}, v_{2,n}) - T(v_{1,n}, v_{2,n})$ for any n, which proves that $(\widehat{v}_1 - \widehat{v}_2) P_1(\widehat{v}_1, \widehat{v}_2) - T(\widehat{v}_1, \widehat{v}_2) \geq (\widehat{v}_1 - \widehat{v}_2) P_1(d) - T(d)$.

12) One might remark that my model is linked to the literature on countervailing incentives or type-dependent participation constraints (see Jullien [2000], Lewis and Sappington [1989] and Maggi and Rodriguez-Clare [1995]). Indeed, the participation constraint of our problem can be written as $dP_1(d) - T(d) \geq v_2$, and v_2 and d are correlated.

13) Note that all the P_1 functions which are constant on $[-t, t]$ are also optimal.

14) Depending on the density function f_Δ, i may be equal to 1 or 2.

15) Note that the usual way to fight collusion is to exclude some of the buyers' types from the auction, which is another form of allocational inefficiency.

16) With $i = 1$ if $\int_s^t L(\delta) f_\Delta(\delta) d\delta \geq 0$, and $i = 2$ otherwise.

17) This is just to ensure that a cartel could not choose which buyer gets the item while paying the minimum price $\underline{v}$. As $r_n = \underline{v}$, the item will always be sold.