An exact vectorial spectral representation of the wave equation for propagation over a Terrain in 3D

Alexandre Chabory, Christophe Morlaas-Courties, Rémi Douvenot

An Exact Vectorial Spectral Representation of the Wave Equation for Propagation Over a Terrain in 3D

A. Chabory, C. Morlaas, and R. Douvenot *

Abstract — **A spectral representation for the electromagntic fields is developed in 3D in cylindrical coordinates for propagation over a Terrain. The result is obtained for a planar terrain and for an homogeneous atmosphere and ground. The spectral representation is related with both the vertical and azimuthal variables. For the vertical variable, this spectral representation contains a discrete component associated with a possible surface wave, and a continuous component. For the azimuthal variable, the spectral representation amounts to a Fourier series.**

1 INTRODUCTION

Split-step methods based on the parabolic approximation are commonly used for the modeling of electromagnetic wave propagation over the ground at large distances [1]. In order to evaluate the field iteratively at larger and larger distances, the computation is realized going back and forth from a spatial to a spectral representation of the wave upon assuming a rotational symmetry about the vertical axis (2D case). The relief, a possible ground wave, and the electrical characteristics of the atmosphere can be taken into account with this method [1].

If the reflection over the ground is modeled by means of a constant surface impedance, the spectral transform corresponds to a continuous mixed Fourier transform [2]. A discretized counterpart of this transform, the discrete mixed Fourier transform (DMFT), has been developed to render the scheme self-consistent and avoid numerical instabilities. In [3, 4], more general ground conditions are considered with non-constant impedance conditions. The method in [4] is based on an exact spectral representation of the vertical operator above a dielectric ground. This spectral representation has been rendered consistent with numerical computation by considering a domain of finite high. An application of this method has been proposed in [5].

However, 2D models are limited in terms of accuracy, notably because they neglect lateral effects, such as reflections over hills that are not aligned with the transmitter-receiver. An extension of split-step methods based on the parabolic equation to three-dimensional configurations has been developed in [6]. The method has been proposed in cartesian coordinates for a ground modeled by a constant surface impedance.

In this article, we consider the propagation above a dielectric ground in 3D, i.e. without assuming a rotational symmetry about the vertical axis. We theoretically develop an exact spectral representation starting from Maxwell equations in cylindrical coordinates. This involves the diagonalization of both the azimuthal and vertical operators.

In Section 2, the problem is expressed in cylindrical coordinates by means of Hertz potentials. In the propagation equation, the operator acting on the vertical coordinate is isolated. In Section 3, this operator is studied as a Sturm-Liouville problem of the third kind [7]. In Section 4, the azimuthal spectral repesentation is introduced using the periodicity of the solution with respect to the azimuth. This yields the spectral representation, which is suitable for the development of a 3D split-step method.

2 FORMULATION

2.1 Configuration

We consider the propagation of a time-harmonic electromagnetic field in the atmosphere taking into account the presence of the ground. We use the cylindrical coordinate system (r, ϕ, z), with unit vectors $(\hat{\boldsymbol{r}}, \hat{\boldsymbol{\phi}}, \hat{\boldsymbol{z}})$, and where z is the vertical axis. The ground/atmosphere are characterized by a constant permeability μ_0 and by space-varying permittivity $\varepsilon_r(r, \phi, z)$ and conductivity $\sigma(r, \phi, z)$. For the spectral representation, the fields are decomposed in two components, viz. one transverse electric (TE) and one transverse magnetic (TM) components with respect to z.

2.2 TM case

The TM case can be formulated via a vector potential $\boldsymbol{\Pi}_{\mathrm{e}}$ oriented along z such that

$$\begin{aligned} \boldsymbol{E} &= \boldsymbol{\Pi}_{\mathrm{e}} - \boldsymbol{\nabla} \frac{1}{k^2} \boldsymbol{\nabla} \cdot \boldsymbol{\Pi}_{\mathrm{e}}, \\ \boldsymbol{H} &= \frac{1}{-j\omega\mu_0} \nabla \times \boldsymbol{\Pi}_{\mathrm{e}}. \end{aligned} \tag{1}$$

where $k = -j\omega\mu_0(\sigma + j\omega\varepsilon)$ is the wavenumber. For an homogeneous atmosphere and ground, and for

*ENAC, TELECOM-EMA, F-31055 Toulouse, France, Toulouse University, F-31400 Toulouse, France, e-mail: alexandre.chabory@recherche.enac.fr.

a ground located at $z=0$, upon replacing $\boldsymbol{\Pi}_{\rm e}$ by $\Psi\hat{\boldsymbol{z}}$, the tranverse components of the electromagnetic fields are given by

$$\begin{aligned} \boldsymbol{E}_{\rm t} &= \frac{1}{k^2}\left(\frac{\partial^2\Psi}{\partial z\partial r}\hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial^2\Psi}{\partial z\partial\phi}\hat{\boldsymbol{\phi}}\right), \\ \boldsymbol{H}_{\rm t} &= -j\omega\mu_0\left(\frac{1}{r}\frac{\partial\Psi}{\partial\phi}\hat{\boldsymbol{r}} - \frac{\partial\Psi}{\partial r}\hat{\boldsymbol{\phi}}\right). \end{aligned} \tag{2}$$

¿From Maxwell equations, the problem can then be reduced to the scalar propagation equation in cylindrical coordinates

$$-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\Psi}{\partial r} - \frac{1}{r^2}\frac{\partial^2\Psi}{\partial\phi^2} - \frac{\partial^2\Psi}{\partial z^2} - k^2(z)\Psi = 0, \tag{3}$$

with $r\in[0,\infty[$, $\phi\in[0,2\pi[$, and $z\in\mathbb{R}$. Besides,

$$k(z) = \begin{cases} k_{\rm a} & z>0, \\ k_{\rm g} & z<0, \end{cases} \tag{4}$$

with $k_{\rm a}$ and $k_{\rm g}$ the wavenumbers in the atmosphere and ground, respectively. At infinity, radiating boundary conditions are imposed on Ψ. At the ground level $z=0$, boundary conditions imposed by the ground/atmosphere interface are the continuity of the tangential components of the electric and magnetic fields at $z=0$. This can be written as

$$\begin{aligned} \frac{1}{k_{\rm a}^2}\left.\frac{\partial\Psi}{\partial z}\right|_{z=0^+} - \frac{1}{k_{\rm g}^2}\left.\frac{\partial\Psi}{\partial z}\right|_{z=0^-} &= 0, \\ \Psi|_{z=0^+} - \Psi|_{z=0^-} &= 0. \end{aligned} \tag{5}$$

For the spectral representation the propagation equation (3) is split into two terms

$$L_{r\phi}\Psi + L_z\Psi = 0, \tag{6}$$

with

$$\begin{aligned} L_{r\phi} &= -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} - \frac{1}{r^2}\frac{\partial^2}{\partial\phi^2}, \\ L_z &= -\frac{\partial^2}{\partial z^2} - k^2. \end{aligned} \tag{7}$$

The operator L_z is only related to the vertical coordinate z, while $L_{r\phi}$ acts on both r and ϕ. The spectral representation is obtained by diagonalizing separately these operators.

2.3 TE case

In a similar way, the TE case can be formulated by means of a vector potential $\boldsymbol{\Pi}_{\rm m}$ oriented along z. For the sake of concision, we will not present the TE case in this article but the derivations are similar to the TM case.

3 VERTICAL SPECTRAL REPRESENTATION

The vertical spectral representation is obtained by means of the diagonalization of the operator L_z. To do so, we use the method presented in [7]. Besides, the demonstration exactly follows the one achieved for the 2D case in [4]. Only the main results are presented here.

Firstly, the Green's function $G(z,z',\lambda)$ of the operator $L_z-\lambda I$ is determined, for $\lambda\in\mathbb{C}$, I the identity operator, and (z',z) the position of the source and observation, respectively. Using a classical method for the determination of Green's functions [7], we obtain

$$G(z,z',\lambda) = \frac{e^{-jk_{z\rm a}|z-z'|} + \Gamma e^{-jk_{z\rm a}(z+z')}}{2jk_{z\rm a}}, \tag{8}$$

for $z\geq 0$ and $z'\geq 0$, with $k_{z\rm a}=\sqrt{k_{\rm a}^2+\lambda}$. The suitable determination of the square root is the one that respects the radiation condition, i.e. $\text{Im}(k_{z\rm a})\leq 0$. Note that we have restricted the computation to the atmosphere, i.e. to $z,z'\geq 0$. To do so, the ground boundary condition is taken into account by means of the reflection coefficient Γ given by

$$\Gamma = \frac{Z_{\rm a}-Z_{\rm g}}{Z_{\rm a}+Z_{\rm g}}, \tag{9}$$

with

$$Z_{\rm a} = \frac{jk_{z\rm a}}{\sigma_{\rm a}+j\omega\varepsilon_{\rm a}}, \quad Z_{\rm g} = \frac{jk_{z\rm g}}{\sigma_{\rm g}+j\omega\varepsilon_{\rm g}}. \tag{10}$$

Secondly, we write the following identity, demonstrated in [7],

$$\frac{1}{2j\pi}\lim_{R\to\infty}\oint_{\mathcal{C}_R} G(z,z',\lambda)d\lambda = -\delta(z-z'), \tag{11}$$

where $\mathcal{C}_R$ is the circle centered at 0 of radius R in the complex λ-plane. Then, we evaluate explicitly the integral in (11). Taking care of the contribution of the pole in Γ and of the branch cut due to $k_{z\rm a}$, we obtain

$$\begin{aligned} \delta(z-z') = {} & \frac{2jk_{z\rm a}^p}{1-\epsilon^2}e^{-jk_{z\rm a}^p(z+z')} \\ & + \frac{1}{2\pi}\int_0^{+\infty}\frac{1}{\Gamma}\left(e^{jk_{z\rm a}z'} + \Gamma e^{-jk_{z\rm a}z'}\right) \\ & .\left(e^{jk_{z\rm a}z} + \Gamma e^{-jk_{z\rm a}z}\right)dk_{z\rm a}. \end{aligned} \tag{12}$$

In this expression, the first term accounts for the pole contribution with

$$k_{z\rm a}^p = \sqrt{k_a^2\frac{\epsilon}{1+\epsilon}}. \tag{13}$$

The second term accounts for the branch cut contribution.

Finally, we write

$$\Psi(r,\phi,z) = \int_0^{\infty} \delta(z-z')\Psi(r,\phi,z')dz'. \quad (14)$$

Using (12), we can substitute $\delta(z-z')$ by the explicit contributions of the branch cut and pole. The expression that we obtain is the sprectral representation of the operator, which is given by the transform pairs

$$\Psi(r,\phi,z) = \tilde{\Psi}^p(r,\phi)e^{-jk_{za}^p z} + \int_0^{+\infty} \tilde{\Psi}(r,\phi,k_z)\left(e^{jk_z z} + \Gamma(k_z)e^{-jk_z z}\right)dk_z, \quad (15)$$

where $\tilde{\Psi}^p(r,\phi)$ and $\tilde{\Psi}(r,\phi)$ are the vertical sprectral components that can be obtained from

$$\begin{aligned} \tilde{\Psi}^p(r) &= \frac{2jk_{za}^p}{1-\epsilon^2}\int_0^{+\infty} \Psi(r,z)e^{-jk_{za}^p z}dz, \\ \tilde{\Psi}(r,k_{za}) &= \frac{1}{2\pi}\int_0^{+\infty} \frac{\Psi(r,z)}{\Gamma(k_{za})} \\ &\quad .\left(e^{jk_{za}z} + \Gamma(k_{za})e^{-jk_{za}z}\right)dz. \end{aligned} \quad (16)$$

The first term of (15) correponds to the contribution of the pole, and can be associated with a possible ground/surface wave. The second term is the continuous spectrum that represents plane waves and their reflection over the ground.

4 AZIMUTHAL SPECTRAL REPRESENTATION

The previous vertical spectral representation (15) can be introduced in (3). This yields

$$-\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial \tilde{\Psi}}{\partial r} - \frac{1}{r^2}\frac{\partial^2 \tilde{\Psi}}{\partial \phi^2} - (k_{\mathrm{a}}^2 - k_{z\mathrm{a}}^2)\tilde{\Psi} = 0. \quad (17)$$

Our aim is to obtain a spectral representation suitable for split-step algorithm, where Ψ is computed iteratively at increasing distances r. Thus we need to add a spectral representation associated with the variable ϕ. This obviously corresponds to a Fourier series because Ψ is necessarily 2π-periodic with respect to ϕ. The spectral representation associated with both vertical and azimuthal variables is finally given by

$$\Psi(r,\phi,z) = \sum_{n_\phi \in \mathbb{Z}} e^{jn_\phi \phi}\left[\hat{\Psi}_{n_\phi}^p(r)e^{-jk_{za}^p z} + \int_0^{+\infty} \hat{\Psi}_{n_\phi}(r,k_z)\left(e^{jk_z z} + \Gamma(k_z)e^{-jk_z z}\right)dk_z\right], \quad (18)$$

where the vertical and azimuthal spectral components are given by

$$\begin{aligned} \hat{\Psi}_{n_\phi}^p(r) =& \frac{2jk_{za}^p}{1-\epsilon^2}\frac{1}{2\pi}\int_0^{2\pi}\int_0^{+\infty} \Psi(r,\phi,z) \\ & .e^{-jk_{za}^p z - jn_\phi \phi}dzd\phi, \\ \hat{\Psi}_{n_\phi}(r,k_{za}) =& \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{+\infty} \frac{\Psi(r,z)}{\Gamma(k_{za})}e^{-jn_\phi \phi} \\ & \left(e^{jk_{za}z} + \Gamma(k_{za})e^{-jk_{za}z}\right)dzd\phi. \end{aligned} \quad (19)$$

5 APPLICATION TO PROPAGATION SIMULATIONS

In this section, we explain how the spectral representation defined in the previous section can be used to express the propagation of a wave in an homogeneous atmosphere above a planar homogeneous ground in 3D. No details are presented about the numerical aspects of the implementation. We assume that Ψ is known at a distance r_0 and that the wave is propagating towards $r \to +\infty$. From (19), the vertical and azimuthal spectral components $\hat{\Psi}_{n_\phi}^p(r_0)$ and $\hat{\Psi}_{n_\phi}(r_0,k_{za})$ can be computed. To predict the wave at $r > r_0$, we explicit the propagation equation in the spectral domain

$$-\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\hat{\Psi}\right) + \frac{n_\phi^2}{r^2}\hat{\Psi} - (k_{\mathrm{a}}^2 - k_{z\mathrm{a}}^2)\hat{\Psi} = 0 \quad (20)$$

Because the field is assumed to propagate towards $r > r_0$, this equation can be solved to obtain the spectrum at $r > r_0$. We end up with

$$\hat{\Psi}(r,k_z) = \hat{\Psi}(r_0,z)\frac{\mathrm{H}_{n_\phi}^{(2)}(k_r r)}{\mathrm{H}_{n_\phi}^{(2)}(k_r r_0)}, \quad (21)$$

with $\mathrm{H}_{n_\phi}^{(2)}$ the Hankel function of the second kind and of order n_ϕ, and $k_r = \sqrt{k_{\mathrm{a}}^2 - k_{z\mathrm{a}}^2}$ where $\mathrm{Im}(k_r) \leq 0$. Finally, we can go back from the spectral to the spatial representation of Ψ at r by means of the expressiosn (18).

6 CONCLUSION

For an homogeneous atmosphere and ground, and for a planar ground, we have developed a spectral representation in 3D in cylindrical coordinates. The spectral representation is related to both the vertical and azimuthal variables. For the vertical variable, this spectral representation contains a discrete component associated with a possible surface wave, and a continuous component. For the azimuthal variable, the spectral representation amounts to a Fourier series. We have explained how this method could be employed to predict the propagation of a wave in an homogeneous atmosphere above a planar homogeneous ground in 3D. A more general split-step algorithm could be derived, which would take into account the terrain profile, and variations in the electrical characteristics as it has been done in 2D [5], or in [1] for methods based on the parabolic-wave equation.

References

[1] M. Levy, Parabolic Equation Methods for Electromagnetic Wave Propagation, ser. IEE electromagnetic waves series 45, IET, Ed., 2000.

[2] D. G. Dockery and J. R. Kuttler, An improved impedance-boundary algorithm for Fourier split-step solutions of the parabolic wave equation, IEEE trans. Antennas Propag., vol. 44, no. 12, pp. 1592-1599, 1996.

[3] R. Janaswamy, "Radio wave propagation over a nonconstant immitttance plane", Radio Sci., vol. 36, no. 3, pp.387-405, 2001.

[4] A. Chabory, C. Morlaas, R. Douvenot, and B. Souny. An exact spectral representation of the wave equation for propagation over a terrain. ICEAA. 2012, pp. 717-720, 2012.

[5] R. Douvenot, C. Morlaas, A. Chabory, B. Souny, "Matrix split-step resolution for propagation based on an exact spectral formulation", ICEAA. 2012, pp. 562-565, 2012.

[6] R. Janaswamy, "Path loss predictions in the presence of buildings on flat terrain: A 3-D vector parabolic equation approach", IEEE trans. Antennas Propag., vol. 51, no 8, p. 1716-1728, 2003.

[7] D. G. Dudley, Mathematical Foundations for Electromagnetic Theory, IEEE press, New-York, 1994.