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# Trajectory Optimization for Differential Flat Systems

Kahina Louadj<sup>1,2</sup>, Benjamas Panomruttanarug<sup>3</sup>, Alexandre Carlos Brandão Ramos<sup>4</sup>, Felix Mora-Camino<sup>1</sup>

**Abstract**—The purpose of this communication is to investigate the applicability of Variational Calculus to the optimization of the operation of differentially flat systems. After introducing characteristic properties of differentially flat systems, the applicability of variational calculus to the optimization of flat output trajectories is displayed. Two illustrative examples are also presented.

**Index Terms**—Differential flatness, Variational Calculus, Trajectory optimization.

## I. INTRODUCTION

IN the last decade a large interest has risen for new non linear control approaches such as non linear inverse control [1,2,3], backstepping control [4] and differential flat control [2]. These control law design approaches present strong similarities. Many dynamical systems have been found to be differentially flat and flat outputs trajectory control has been in general performed using non linear inverse control, called in that case differential flat control. This approach assumes that a flat outputs reference trajectory is already available. However, this is not the case in many situations. So the problem of designing an optimal flat outputs trajectory should be considered.

In this paper it is showed that variational calculus and more specially Euler Equation can provide a solution to this problem without having to consider the intricacies associated with the application of the Minimum Principle of Pontryaguine or the Hamilton-Jacobi-Bellman equations. Two illustrative examples are deployed.

## II. DIFFERENTIAL FLAT OUTPUT AND CONTROL

Consider a general non-linear dynamic continuous system given by:

$$\dot{\underline{X}} = f(\underline{X}, \underline{U}) \quad (1)$$

$$\underline{Y} = h(\underline{X}) \quad (2)$$

where  $\underline{X} \in \mathbf{R}^n$  is the state vector,  $\underline{U} \in \mathbf{R}^m$  is the control vector,  $\underline{Y} \in \mathbf{R}^m$ ,  $f$  is a smooth vector field of  $\underline{X}$  and  $\underline{U}$  and  $h$  is a smooth vector field of  $\underline{X}$ .

It is supposed here that each input has an independent effect on the state dynamics:

$$\text{rank}[\partial f / \partial u, \dots, \partial f / \partial u_m] = m \quad (3)$$

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### A. Relative Degrees of Outputs in Nonlinear Systems

According to [1] the system (1)-(2) is said to have with respect to each independent output  $Y_i$ , a relative degree  $r_i$  if the output dynamics can be written as:

$$\begin{pmatrix} Y_1^{(r_1+1)} \\ \vdots \\ Y_m^{(r_m+1)} \end{pmatrix} = \begin{pmatrix} b_1(\underline{X}, \underline{U}) \\ \vdots \\ b_m(\underline{X}, \underline{U}) \end{pmatrix} \quad (4)$$

with

$$Y_i^{(s)} = a_{js}(\underline{X}) \quad s = 0, \dots, r_j, j = 1, \dots, m \quad (5)$$

and

$$\partial b_j(\underline{X}, \underline{U}) / \partial \underline{U} \neq 0 \quad j = 1, \dots, m \quad (6)$$

The output dynamics (4)-(5) can be rewritten globally as:

$$\underline{Z} = A(\underline{X}) \quad (7)$$

and

$$\tilde{\underline{Z}} = B(\underline{X}, \underline{U}) \quad (8)$$

where

$$\underline{Z} = (Y_1 \dots Y_1^{(r_1)} \dots Y_m \dots Y_m^{(r_m)})' \quad (9)$$

and

$$\tilde{\underline{Z}} = (Y_1^{(r_1+1)}, \dots, Y_m^{(r_m+1)})' \quad (10)$$

Here

$$\underline{A}(\underline{X}) = \begin{pmatrix} a_1(\underline{X}) \\ \vdots \\ a_m(\underline{X}) \end{pmatrix} \quad (11)$$

with

$$\underline{a}_j(\underline{X}) = \begin{pmatrix} a_{j0}(\underline{X}) \\ \vdots \\ a_{j,r_j}(\underline{X}) \end{pmatrix} \quad j = 1, \dots, m \quad (12)$$

The relative degrees obey (see [2]) to the condition:

$$\sum_{i=1}^m (r_i + 1) \leq n, \quad i = 1, \dots, m \quad (13)$$

When the strict equality holds, vector  $\tilde{\underline{Z}}$  can be adopted as a new state vector for system (1), otherwise internal dynamics must be considered.

From (8), while  $B(\underline{X}, \underline{U})$  is invertible with respect to  $\underline{U}$ , an output feedback control law such as:

$$\underline{U}(\underline{X}) = B_u^{-1}(\underline{X}) \tilde{\underline{Z}} \quad (14)$$

can be adopted.

### B. Differential Flat System

Now suppose that  $\underline{Y} \in \mathbf{R}^m$  is a differential flat output for system (1), then from [3] the state and the input vectors can be written as:

$$\underline{X} = \eta(\underline{Z}) \quad (15)$$

$$\underline{U} = \xi(\underline{Z}, \dot{\underline{Z}}) \quad (16)$$

where  $\underline{Z}$ ,  $\dot{\underline{Z}}$  are given respectively by (9) and (10). Here  $\eta(\cdot)$  is a function of  $Y_j$  and its derivatives up to order  $r_j$ , and  $\xi(\cdot)$  is a function of  $Z_j$  and its derivatives up to order  $r_{j+1}$ , for  $j = 1$  to  $m$  where the  $r_j$  are integers.

It appears of interest to introduce here three new definitions. The differential flat system is said output observable if:

$$\text{rank}([\partial\eta/\partial\underline{Z}]) = n \quad (17)$$

The differential flat system is said full flat differential if:

$$\sum_{i=1}^m r_i = n - m \quad (18)$$

The differential flat system (1) is said output controllable if:

$$\det([\partial\xi/\partial\dot{\underline{Z}}]) \neq 0 \quad (19)$$

In that case too, it is easy to derive a control law of order  $r_{j+1}$  with respect to output  $j$  by considering an output dynamics such as:

$$\dot{\underline{Z}} = C(\underline{Z}, \underline{V}) \quad (20)$$

where  $\underline{V} \in \mathbf{R}^m$  is an independent input, since then:

$$\underline{U} = \xi(\underline{Z}, C(\underline{Z}, \underline{V})) \quad (21)$$

### C. Flatness and Internal Dynamics

It appears from relations (7) and (8) that a sufficient condition for system (1) to be differentially flat output observable and output controllable with respect to  $\underline{Y}$  given by (2) is that  $A$  is invertible with respect to  $\underline{X}$  and that  $B$  is invertible with respect to  $\underline{U}$ .

A necessary condition for the invertibility of  $A$  is:

$$\sum_{i=1}^m r_i = n - m \quad (22)$$

while (3) is a necessary condition for the invertibility of  $B$  with respect to  $\underline{U}$ . In that case it is possible to define function  $\eta$  and  $\xi$  by:

$$\underline{X} = A^{-1}(\underline{Z}) = \eta(\underline{Y}, \dot{\underline{Y}}, \dots, \underline{Y}^{(p)}) \quad (23)$$

and

$$\underline{U} = B_u^{-1}(A^{-1}(\underline{Z}))(\underline{Z}) = \xi(\underline{Y}, \dot{\underline{Y}}, \dots, \underline{Y}^{(p+1)}) \quad (24)$$

Here:

$$p = \max r_j, \quad j = 1 \text{ to } m \quad (25)$$

Then, a sufficient condition for differential flatness of  $\underline{Z}$  is that  $\underline{Z}$  is a state vector for system (1), i.e. there are no internal dynamics in this case.

## III. OPTIMAL CONTROL OF DIFFERENTIALLY FLAT SYSTEMS

Here the system (1), (2) is assumed to the differentially flat with respect to  $\underline{Y}$ , so that relation (23) and (24) hold.

### A. Formulation of the Considered Optimal Control Problems

Here can be considered optimization criteria over a given span of time  $[0, T]$  such as:

$$\min_{(\underline{X}, \underline{U}(t))} F(\underline{X}(T)) + \int_0^T g(\underline{X}, \underline{U}(t)) dt \quad (26)$$

or when the focus is on the trajectory developed by the differentially flat outputs:

$$\min_{(\underline{Y}, \underline{U}(t))} F(\underline{Y}(T)) + \int_0^T g(\underline{Y}, \underline{U}(t)) dt \quad (27)$$

Let us consider vector  $\underline{Z}$  given by (24), in both cases, using relation (23) and (24) the optimization criteria can be written under the form:

$$\min_{\underline{Z}} F(\underline{Z}(T)) + \int_0^T \psi(\underline{Z}, \dot{\underline{Z}}) dt \quad (28)$$

where  $\underline{Z}(T)$  must satisfy partial constraints at time 0 and  $T$ , determined from the initial and final constraints on the state or the outputs.

### B. Variational Calculus Solution

Consider that the optimal control problem built from relation (26) or (27) with (1) and (2) does not consider explicitly the state equation. Since it can be rewritten :

$$\dot{\underline{Z}} = \mathbf{C}\dot{\underline{Z}} \quad (29)$$

where  $\mathbf{C}$  a 0-1 matrix with a single 1, by row. Here is introduced an auxiliary function  $\varphi$  given by:

$$\varphi(\underline{Z}, \dot{\underline{Z}}) = \Psi(\underline{Z}, \dot{\underline{Z}}) + \lambda^t (\mathbf{C}\dot{\underline{Z}} - \dot{\underline{Z}}). \quad (30)$$

Then, problem (28) terms out to be a classical variational calculus problem to which Euler's equation will provide necessary optimality conditions. Here the Euler equations are given by:

$$\frac{\partial\varphi}{\partial\underline{Z}} - \frac{d}{dt} \left( \frac{\partial\varphi}{\partial\dot{\underline{Z}}} \right) = 0 \quad (31)$$

and

$$\frac{\partial\varphi}{\partial\dot{\underline{Z}}} - \frac{d}{dt} \left( \frac{\partial\varphi}{\partial\ddot{\underline{Z}}} \right) = 0 \quad (32)$$

Let  $\underline{Z}^*$  be the solution satisfying (31), (32) with the initial and final constraints. Then the solution of the original problem will be:

$$\underline{X}^* = \eta(\underline{Z}^*) \quad \text{such} \quad \underline{U}^* = \xi(\underline{Z}^*, \dot{\underline{Z}}^*) \quad (33)$$

## IV. EXAMPLES

### A. Example 1

Consider the optimization problem

$$\text{Min}_u \int_0^T u^2 dt \quad (34)$$

with the linear state equations:

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = u, \end{cases} \quad y = x_1 \quad (35)$$

with the limit conditions

$$x_1(0) = 0, \quad x_1(T) = 1, \quad x_2(0) = 0, \quad x_2(T) = 0 \quad (36)$$

From (35) it is clear that  $y$  is a differentially flat output with:

$$u = \ddot{y} \quad (37)$$

and

$$x_1 = y, \quad x_2 = \dot{y} \quad (38)$$

and introducing

$$Z_1 = x_1 \quad \text{and} \quad Z_2 = \dot{Z}_1, \quad (39)$$

the optimal control problem can be rewritten as:

$$\text{Min} \int_0^T \dot{Z}_2^2 dt \quad (40)$$

with

$$\dot{Z}_1 - Z_2 = 0 \quad (41)$$

then introducing the auxiliary function:

$$\varphi = \dot{Z}_2^2 + \lambda(\dot{Z}_1 - Z_2) \quad (42)$$

where  $\lambda$  is a parameter, the Euler equations are such that:

$$\frac{\partial \varphi}{\partial Z_1} - \frac{d}{dt} \left( \frac{\partial \varphi}{\partial \dot{Z}_1} \right) = 0 \quad (43)$$

$$\frac{\partial \varphi}{\partial Z_2} - \frac{d}{dt} \left( \frac{\partial \varphi}{\partial \dot{Z}_2} \right) = 0 \quad (44)$$

From (43), we get:

$$-\dot{\lambda} = 0 \Rightarrow \lambda = cst \quad (45)$$

From (44), we get:

$$-\lambda - \ddot{Z}_2 = 0 \Rightarrow \ddot{Z}_2 = -1/2\lambda = cst = c \quad (46)$$

Then,

$$\dot{Z}_2 = c_0 + ct \Rightarrow Z_2 = c_1 + c_0t + 1/2ct^2 \quad (47)$$

From (41), we obtain:

$$Z_1 = c_2 + c_1t + 1/2c_0t^2 + 1/6ct^3 \quad (48)$$

The constants  $c$ ,  $c_0$ ,  $c_1$  are determined by the limit constraints (36):

$$c_2 = 0, \quad c_1 = 0, \quad c = \frac{-6}{T^3}, \quad c_0 = \frac{3}{T^2} \quad (49)$$

The optimal solution is such as:

$$y^* = x_1^* = \frac{3}{2T^2}t - \frac{1}{T^3}t^2 \quad (50)$$

with

$$u^* = \frac{3}{T^3} - \frac{6}{T^3}t \quad (51)$$

## B. Example 2

Consider the optimization problem

$$\text{Min}_u \int_0^T u^2 dt \quad (52)$$

with the nonlinear state equations:

$$\begin{cases} \dot{x}_1 = x_2^2, \\ \dot{x}_2 = u, \end{cases} \quad y = x_1 \quad (53)$$

with the limit conditions

$$x_1(0) = 0, \quad x_1(T) = 1, \quad x_2(0) = 0, \quad x_2(T) = 0 \quad (54)$$

From (53) it is clear that  $y$  is a differentially flat output with:

$$u = \frac{1}{2\sqrt{\dot{y}}} \cdot \ddot{y} \quad (55)$$

and

$$x_1 = y, \quad x_2 = \sqrt{\dot{y}} \quad (56)$$

and introducing

$$Z_1 = x_1, \quad Z_2 = x_2, \quad (57)$$

the optimal control problem can be rewritten as:

$$\text{Min} \int_0^T \dot{Z}_2^2 dt \quad (58)$$

with

$$\dot{Z}_1 - Z_2^2 = 0 \quad (59)$$

then introducing the auxiliary function:

$$\varphi = \dot{Z}_2^2 + \lambda(\dot{Z}_1 - Z_2^2) \quad (60)$$

where  $\lambda$  is a parameter, the Euler-Lagrange equations are such that:

$$\frac{\partial \varphi}{\partial Z_1} - \frac{d}{dt} \left( \frac{\partial \varphi}{\partial \dot{Z}_1} \right) = 0 \quad (61)$$

$$\frac{\partial \varphi}{\partial Z_2} - \frac{d}{dt} \left( \frac{\partial \varphi}{\partial \dot{Z}_2} \right) = 0 \quad (62)$$

From (61), we get:

$$-\dot{\lambda} = 0 \Rightarrow \lambda = cst \quad (63)$$

From (62), we get:

$$-2\lambda Z_2 - 2\ddot{Z}_2 = 0 \Rightarrow \lambda Z_2 + \ddot{Z}_2 = 0. \quad (64)$$

When supposing that  $\lambda$  is negative, it appears that the resulting solution cannot satisfy limit conditions (54), then, here is considered a solution of (64) when  $\lambda$  is taken positive:

$$Z_2 = \alpha e^{(jt\sqrt{\lambda})} + \beta e^{(-jt\sqrt{\lambda})} \quad (65)$$

and the optimal solution is given by:

$$y^* = x_1^* = \frac{1}{2T\pi} \sin\left(\frac{4\pi}{T}t\right) \quad (66)$$

with

$$u^* = \frac{4\pi}{T^{\frac{3}{2}}} \cos\left(\frac{2\pi}{T}t\right) \quad (67)$$

## V. CONCLUSION

From the above examples it appears that it is worth to consider the differential flatness property when it exists to solve trajectory optimization problems. In both cases, the optimal solutions have been found analytically, however in other cases, a numerical solution should be pursued. This line on research will be pursued considering input constraints in the optimization problem.

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