4D Trajectories Complexity Metric Based on Lyapunov Exponents

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The ATM system has to cope with an increasing number of flights, pushing the capacity to its limits. As an example, the average daily traffic above Europe was 26286 flights/day, with a peak traffic demand in excess of 31000 flights [11]. Although delays are kept low, it is expected from the same reference that capacity has to be extended in the future. Basically, two strategies can be devised : adapt the demand to capacity (slot-route allocation, collaborative decision making, . . .) or adapt the capacity to the demand (Airspace design, 4D trajectory planning, autonomous aircraft, . . .). The first approach can be used in the context of current ATM system, while innovative future designs will mainly follow the second strategy. Currently, complexity of the traffic is measured only as an operational capacity : the maximum number of aircraft that ATC controllers are willing to accept is fixed on a per-sector basis and complexity is assessed by comparing the real number of aircraft with the sector capacity. It must be noted that under some circumstances, controllers will accept aircraft beyond the capacity threshold while rejecting traffic at other times although the number of aircraft is well below the maximum capacity. This simple fact clearly show that capacity as a crude complexity metric is not enough by itself to fully account for the controller's workload. In order to better quantify the complexity, geometric features of the traffic have to be included. As previously stated, depending on the traffic structure, ATC controllers will perceive differently situations, even if the number of aircraft present in the sector is the same. Furthermore, exogenous parameters like the workload history can be influential on the perceived complexity at a given time (a long period of heavy load will tend to reduce the efficiency of a controller). Some reviews of complexity in ATC have been completed, mainly from the controller's workload point of view [4], [10], and have recognized that complexity is related to both the structure of the traffic and the geometry of the airspace. This tends to prove that controller's workload has two facets :

- An intrinsic complexity related to traffic structure.
- A human factor aspect related to the controller itself.

While most complexity metrics tend to capture those effects within a single aggregate indicator, the purpose of this work is to design a measure of intrinsic complexity only since it is the most relevant metric for an highly automated ATC system (no human factors). The first complexity indicator incorporating structural considerations along with the simple number of aircraft is the "Dynamic Density" of Laudeman et al. from NASA [9]. The "Dynamic Density" is a weighted sum of the traffic density (number of aircraft), the number of heading changes (> 15 degrees), the number of speed changes (>0.02 Mach), the number of altitude changes (>750 ft), the number of aircraft with 3-D Euclidean distance between 0-25 nautical miles, the number of conflicts predicted in 25-40 nautical miles. These factors are summed together using weighting factors that were determined by showing different traffic scenarios to several controllers. B.Sridhar from NASA [12], has developed a model to predict the evolution of such a metric in the near future. Efforts to define "Dynamic Density" have identified the importance of a wide range of potential complexity factors, including structural considerations. However, the instantaneous position and speeds of the traffic itself does not appear to be enough to describe the total complexity associated with an airspace. A few previous studies have attempted to include structural consideration in complexity metrics, but have done so only to a restricted degree. For example, the Wyndemere Corporation proposed a metric that included a term based on the relationship between aircraft headings and dominant geometric axis in a sector [7]. The importance of including structural consideration has been explicitly identified in work at Eurocontrol. In a study to identify complexity factors using judgment analysis, Airspace Design was identified as the second most important factor behind traffic volume [8]. Histon, et, al. [5], [6] investigated how this structure can be used to support structure-based abstractions that controllers appear to use to simplify traffic situations. The previous models do not take into account the intrinsic traffic disorder which is related to the complexity. The first efforts related with disorder can be found in [2]. This paper introduces two classes of metrics which measure the disorder of a traffic pattern.G.Aigoin has extended and refined the geometrical class by using a cluster based analysis [1]. All the previous metrics capture only one feature of the complexity and are not able to produce an aggregate metric which can capture all the possible situations (high-low density, how-low convergence, translation organization, round about organization etc ...). The work presented in this paper is based on dynamical systems modeling of the air traffic. A dynamical system describes the evolution of a given state vector. If such a vector is given by the position of aircraft $\vec{X} = [x, y, z]^T$, a dynamical system associates a speed vector $\dot{\vec{X}} = [v_x, v_y, v_z]^T$ to each point in the airspace. The key idea is to find a dynamical system which models the observed aircraft trajectories. Based on this

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dynamical system modeling, a trajectory disorder metric can be computed.

A. Dynamical System Modeling

The dynamical system used to model aircraft trajectories is given by the following equation :

$$\dot{\vec{X}}(t) = V(\vec{X}(t), \omega) \tag{1}$$

where $\vec{X}(t)$ is the state vector of the system ($\vec{X} = [x(t), y(t), z(t)]^T$) and $V : C^2$ vector field depending on parameters ω, describe systems which integral curves may fit the observed trajectories.

This equation associates a vector speed $\dot{\vec{X}}$ to a position in the space coordinate $\vec{X}$ and then synthesis a particular vector field.

Based on the observations of the aircraft (positions, speed vectors and times), the dynamical system has to be adjusted with the minimum error.This fitting is done with a Least Square Minimization (LMS) method. For each considered aircraft i, it is supposed that position $\vec{X}_i = [x_i, y_i, z_i]^T$ and speed vector $\vec{V}_i = [v_{x_i}, v_{y_i}, v_{z_i}]^T$ are given (radar tracker data). An error criterion between the dynamical system model and the observation is computed :

$$E = \sum_{i=1}^{i=N} \left\| \vec{V}_i - V(\vec{X}_i(t), \omega) \right\| \tag{2}$$

where N is number of aircraft.

There are many classical ways of obtaining a class of parametrized vector fields which fulfill the fitting requirement. Among them, vector splines allow a control on the smoothness of vector fields, which is important in our case since civil aircraft maneuvers are based on low acceleration guidance laws.

Computing traffic complexity for a given traffic situation requires interpolating a vector field given only samples (positions and speeds of aircraft at a given time). Vector spline interpolation seeks the minimum of a functional of the form :

$$\frac{1}{2}\int_D \|L\vec{V}(\vec{X})\|^2 d\vec{X} + \frac{1}{2}\sum_{i=1}^{m} \|\vec{V}(\vec{X}_i) - \vec{V}_i\|^2 \tag{3}$$

where $\vec{V}$ is a vector field defined on a domain $D \subset \mathbb{R}^n$, L is an elliptic differential operator controlling smoothness of the solution and $(\vec{X}_i, \vec{V}_i)_{i=1\ldots m}$ are the interpolation data [3]. By introducing the adjoint operator L^T, optimal vector field can be shown to be a linear combination of shifted version of the elementary solution kernel of the differential operator $L^T L$. A special case is the so-called "div-curl" splines with the criterion :

$$\int_{R^2} \alpha \|\nabla div\vec{V}(\vec{X})\|^2 + \beta \|\nabla curl\vec{V}(\vec{X})\| d\vec{X} \tag{4}$$

with α, β positive reals controlling the smoothness of the approximation by focusing on constant divergence or constant curl.

The metric chosen for complexity computation relies on a measure of sensitivity to initial conditions of the underlying dynamical system called Lyapunov exponents. In order to figure out what Lyapunov exponents are, let consider a point and look at its evolution when transported by the dynamical system.

Let $\vec{X}$ be fixed (initial point) and let ϕ be a point trajectory of the dynamical system given by :

$$\phi(t, \vec{X}) = \vec{X} + \int_0^t \vec{V}\phi(s, \vec{X}) ds \tag{5}$$

Assume now that trajectory is disturbed by a small perturbation $\vec{\epsilon}$, we have :

$$\phi(t, \vec{X} + \vec{\epsilon}) = \phi(t, \vec{X}) + \mathbf{D}_{\vec{X}}\phi(t, \vec{X}) \cdot \vec{\epsilon} + o(\|\vec{\epsilon}\|) \tag{6}$$

where $\mathbf{D}_{\vec{X}}\phi(t, \vec{X})$ is the differential of the vector field at $\vec{X}$ that satisfies :

$$\frac{\mathbf{D}_{\vec{X}}\phi(t, \vec{X})}{dt} = \mathbf{D}_{\vec{X}}\vec{V}(\phi(t, \vec{X})) \cdot \mathbf{D}_{\vec{X}}\phi(t, \vec{X}) \tag{7}$$

The Lyapunov exponents are closely related to the singular values of the matrix $\mathbf{D}_{\vec{X}}\phi(t, \vec{X})$ and can be thought as local shear values for the dynamical system.

When Lyapunov exponents are high, the trajectory of a point under the action of the dynamical system is very sensitive to initial conditions (or parameters on which the vector field may depend), so that situation in the future is unpredictable. On the other hand, small values of the Lyapunov exponents mean that the future is highly predictable (expected to be comfortable for a controller). *So, the Lyapunov exponent map determines the area where the underlying dynamical system is organized. It identifies the places where the relative distances between aircraft do not change with time (low real value) and the ones where such distance change a lot (hight real value).*

Let us now describe the practical procedure for computing complexity maps.

First of all, the optimal dynamic div-curl approximation for the observed trajectories is computed, based on the defining equations. That step requires a linear system solving.

The second step computes the second derivatives matrix $\mathbf{D}_{\vec{X}}\phi$ at each point of the grid for ϕ trajectory starting at $\vec{X}$. This is done by solving the differential equation 7 with a Runge-Kutta integrator. The complexity value at point $\vec{X}$ is then obtained by averaging Lyapunov exponents over the time :

$$\kappa(\vec{X}) = \frac{1}{n}\sum_{i=1}^{i=n} \|\mathbf{D}_{\vec{X}}\vec{V}(\phi(t, \vec{X}))\|_2 \tag{8}$$

The figure 1 shows an example of Lyapunov exponents map for which full organized miles in trail trajectories (from south west to north east) cross two random traffic situations. This figure shows clearly a complexity valley on the miles in trail direction. This organization may have been detected even if the miles in trail trajectories would have been structured on a curve trajectory. That is the strong point of this metric: **Lyapunov exponents are able to identify any kind of trajectory organization**.

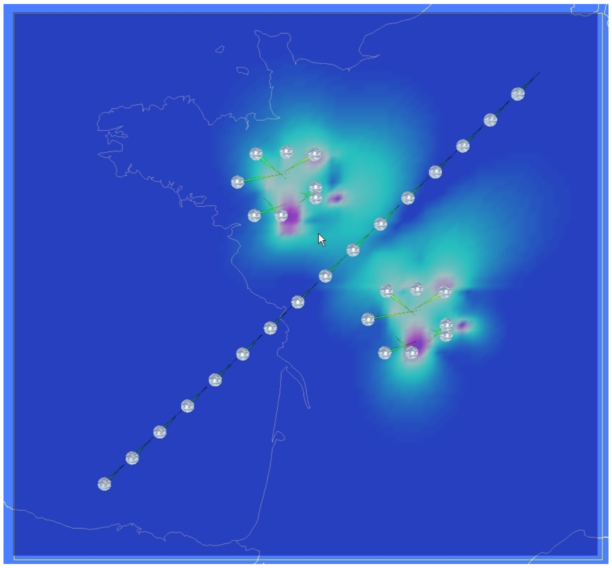

Fig. 1. Lyapunov exponents map. On this figure aircraft are shown with circles with radius equal to the separation norm.

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