Fuzzy Dual Dynamic Programming

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Abstract—**This paper introduces Fuzzy Dual Dynamic Programming as a specific realization of Fuzzy Dynamic Programming to tackle more efficiently optimization problems with fuzzy uncertainty in the values of involved parameters. A case study considering long term airport planning is discussed.**

Keywords-dynamic programming; fuzzy dual numbers; fuzzy dual calculus; airport planning.

I. Introduction

While deterministic optimization problems are formulated with assumed known parameters, very often, real world problems introduces various degrees of uncertainty in problem parameters. When the parameters are only known to remain within given bounds, one way to tackle such problems is through robustness analysis. When probability distributions are available for their values, stochastic optimization techniques may provide the most expected solution. An intermediate approach adopts the fuzzy formalism [1] to represent parameter uncertainties to provide the most possible solution. These three approaches lead in general to cumbersome computations.

This paper adopts the recent formalism of fuzzy dual numbers introduced by Cosenza and Mora-Camino in 2011 [2], to treat parameter uncertainty and solution diversion in mathematical optimization problems through a better trade-off between complexity and effectiveness of the proposed solution. First, fuzzy dual numbers and fuzzy dual calculus are introduced. Then Dynamic Programming is considered, including its general fuzzy version [3], before Fuzzy Dual Dynamic Programming specific characteristics are introduced and discussed. Finally, a case study considering financial risk associated to long term airport planning is developed.

II. Fuzzy Dual Numbers

A. Definition

A set of fuzzy dual numbers is defined as the set $\tilde{\Delta}$ of numbers of the form (*a*, *b*)= $a+\varepsilon.b$, where *a* is the primal part and b is the dual part of the fuzzy dual number $\forall a \in R, \forall b \in R^{+}$. ε represents the unity pure dual number.

A fuzzy dual number loses both its dual and fuzzy attributes if *b* equals zero. The lower and upper bounds of $a+\varepsilon.b$ are given by:

$$B^{low}(a+\varepsilon.b)=a-b \quad \text{and} \quad B^{high}(a+\varepsilon.b)=a+b \tag{1}$$

The *pseudo norm* of a fuzzy dual number is given by $\|a+\varepsilon.b\|=|a|+\rho.b \in R^{+}$, where $\rho>0$ is the shape parameter. The shape parameter is given by:

$$\rho=(1/b)\int_{-b}^{+b}\mu(u)du \tag{2}$$

where μ is the membership function which is supposed to be symmetric with respect to *a*. . Figure 1 shows an example of triangular fuzzy dual membership function.

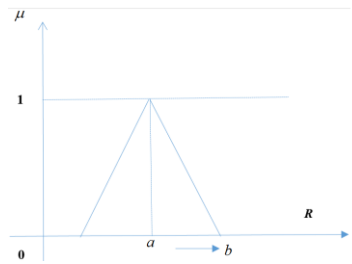

Fig. 1 Graphical representation of a triangular fuzzy dual number

Figure 2 gives several graphical representations of fuzzy dual numbers with different shape parameters.

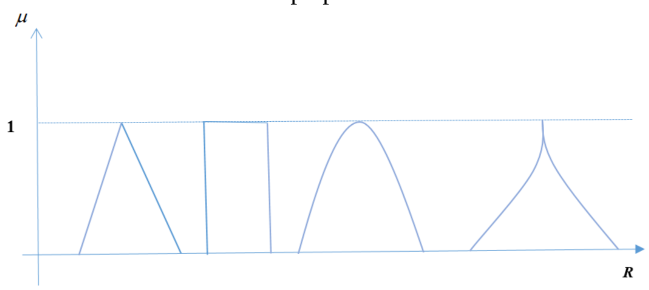

Fig. 2 Examples of fuzzy dual numbers with different shape parameters

The following properties of the pseudo norm are maintained, no matter the values the shape parameters take:

$$\forall a+\varepsilon.b \in \tilde{\Delta} : \|a+\varepsilon.b\| \geq 0 \tag{3}$$

$$\forall a \in R, \forall b \in R^{+} \ \|a+\varepsilon.b\|=0 \Rightarrow a=b=0 \tag{4}$$

$$\|(a+\varepsilon.b)+(\alpha+\varepsilon.\beta)\| \leq \|a+\varepsilon.b\|+\|\alpha+\varepsilon.\beta\|$$

$$\forall a,\alpha \in R, \forall b,\beta \in R^{+} \quad (5)$$
$$\left\| \lambda.(a+\varepsilon.b) \right\| = \lambda.\left\| a+\varepsilon.b \right\|$$

B. *Ordering Fuzzy Dual Numbers*

When comparing two fuzzy dual numbers, only four different relative situations appear. They are represented in figure 3:

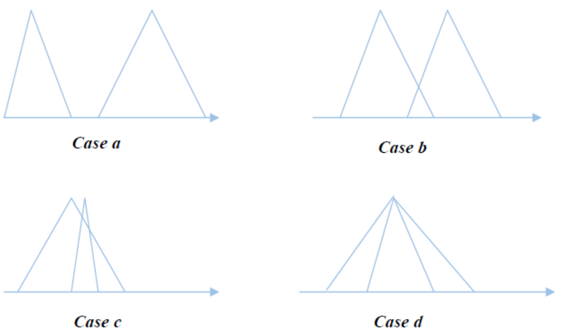

Fig. 3 Relative situations of two fuzzy dual numbers

Here *Case a* corresponds to a *strong* partial order, written $\overset{\smile}{\succ}$, which is be defined over $\tilde{\Delta}$ by:

$$\forall a_1 + \varepsilon b_1, a_2 + \varepsilon b_2 \in \tilde{\Delta} : a_1 + \varepsilon b_1 \overset{\smile}{\succ} a_2 + \varepsilon b_2$$
$$\Leftrightarrow a_1 - \rho b_1 \succ a_2 + \rho b_2 \quad (6)$$

The *mean* partial order of *case b*, written $\hat{\geq}$, is defined over $\tilde{\Delta}$ by:

$$\forall a_1 + \varepsilon b_1, a_2 + \varepsilon b_2 \in \tilde{\Delta} : a_1 + \varepsilon b_1 \overset{\frown}{\succ} a_{\acute{e}} + \varepsilon b_2$$
$$\Leftrightarrow a_1 + \rho b_1 \succ a_2 + \rho b_2 \succ a_1 - \rho b_1 \quad (7)$$

The *weak* partial order of *case c*, written $\tilde{\succ}$, is such as:

$$a_1 > a_2,\ a_1 - \rho b_1 \succ a_2 - \rho b_2,\ a_1 + \rho b_1 \prec a_2 - \rho b_2 \quad (8)$$

The *fuzzy equality* between two fuzzy dual numbers, corresponding to case *d*, is symbolized by $\cong$ and is characterized by:

$$a_1 = a_2 \quad \text{and} \quad b_1 = b_2 \quad (9)$$

Then, it appears that it is *always possible* to rank two fuzzy dual numbers and to assign a qualitative evaluation to this comparison (strong, mean or weak). When either (6), (7) or (8) is satisfied, it will be said that fuzzy dual number $a_1 + \varepsilon b_1$ is greater than fuzzy dual number $a_2 + \varepsilon b_2$ and we will write:

$$a_1 + \varepsilon b_1 \succ a_2 + \varepsilon b_2 \quad (10)$$

A *degree of certainty c* can be attached to this assertion. A candidate expression for this degree is given by:

$$c = 1 - \frac{1}{2}\min\left\{\frac{\alpha}{b_1}, \frac{\alpha}{b_2}\right\} \quad \text{if} \quad a_1 \geq a_2 \quad (11\text{-a})$$

and

$$c = \frac{1}{2}\min\left\{\frac{\alpha}{b_1}, \frac{\alpha}{b_2}\right\} \quad \text{if} \quad a_1 < a_2 \quad (11\text{-b})$$

where α is the area of the intersection between fuzzy dual numbers $a_1 + \varepsilon b_1$ and $a_2 + \varepsilon b_2$.

In Figure 3, in case a: *c*=1, in case b: *c*=0.9, in case c: *c*=0.7 and in case d: *c* =0.5.

III. Fuzzy Dual Calculus

Here basic operations with fuzzy dual numbers are briefly introduced and extended to fuzzy dual vectors and matrices as described in [4].

A. *Basic Operations*

The fuzzy dual neutral element is given by: $\tilde{0} = (0+\varepsilon 0)$. The neutral element of fuzzy dual multiplication is given by $\tilde{1} = (1+\varepsilon 0)$ and only non-zero crisp numbers have an inverse.

The fuzzy dual addition of two fuzzy dual numbers, denoted by $\tilde{+}$, identical to the dual numbers addition, is given by:

$$(a_1 + \varepsilon b_1) \tilde{+} (a_2 + \varepsilon b_2) = (a_1 + a_2) + \varepsilon(b_1 + b_2) \quad (12)$$

The fuzzy dual product of two fuzzy dual umbers, denoted by $\tilde{\bullet}$, is given by:

$$(a_1 + \varepsilon b_1) \tilde{\bullet} (a_2 + \varepsilon b_2) = (a_1 \cdot a_2 + \varepsilon(|a_1| \cdot b_2 + |a_2| b_1)) \quad (13)$$

The fuzzy dual product is constructed in a way that the fuzzy interpretation of the dual part is preserved but is different from dual calculus.

Both fuzzy dual addition and fuzzy dual multiplication are commutative and associative, while fuzzy dual multiplication is also distributive with respect to the fuzzy dual addition. The *nilpotent* property of operator ε is maintained:

$$\varepsilon \tilde{\bullet} \varepsilon = \varepsilon^2 = \tilde{0} \quad (14)$$

B. *Fuzzy dual vectors and matrices*

Let E be a Euclidean space of dimension p over R, we construct a set $\tilde{E}$ composed of pairs of vectors, which are called dual fuzzy vectors taken from the Cartesian product $E \times E^{+}$, where E^{+} is the positive half-space of E in its canonical basis. The following operations are defined over $\tilde{E}$:

Addition:

$$(a,b) + (c,d) = (a+c, b+d) \qquad \forall a,b \in E \quad \forall c,d \in E^{+} \quad (15)$$

The multiplication by a fuzzy dual scalar $\lambda + \varepsilon\ \mu$ is given by:

$$(\lambda + \varepsilon\ \mu)\,(a,b) = (\lambda\, a, |\lambda|\ b + \mu |a|) \quad \forall \lambda + \varepsilon\ \mu \in \tilde{\Delta}, \forall (a,b) \in \tilde{E} \quad (16)$$

The set $\tilde{M}_n$ of fuzzy dual square matrices of order $n \times n$ is constructed on the same logic as fuzzy dual numbers and fuzzy dual vectors. Hence, a fuzzy dual matrix will be defined as:

$$A = [a_{ij}] = [r(a_{ij}) + \varepsilon\, d(a_{ij})] = r(A) + \varepsilon\, d(A) \quad (17)$$

where $r(A)$ is a $R^{n \times n}$ matrix and $d(A)$ is a positive $R^{n \times n}$ matrix.

The basic operations over dual square matrices will be defined as follows:

$$A+B=R(A)+R(B)+\varepsilon\,(D(A)+D(B)) \qquad \forall A,B\in\tilde{M} \quad (18)$$

$$A\bullet B=R(A)\,R(B)+\varepsilon\,(|R(A)|\cdot D(B)+D(A)|R(B)|) \quad \forall A,B\in\tilde{M} \quad (19)$$

$$\lambda\;A=R(\lambda)\;R(A)+\varepsilon\,(|R(\lambda)|\;D(A)+D(\lambda)\;|R(A)|)\;\forall\lambda\in\tilde{\Delta} \quad (20)$$

The product of a fuzzy dual square matrix by a fuzzy dual vector *u* is considered in this context to be a fuzzy dual vector given by:

$$A\times u=R(A)R(u)+\varepsilon\,(|R(A)|\;D(u)+D(A)\;|R(u)|) \quad (21)$$

IV. FUZZY DYNAMIC PROGRAMMING

The Since its publication in the late 1950's by R. Bellman, Dynamic Programming has become very quickly a widely applied mathematical formalism in decision-making processes.

A. *Dynamic Programming*

Dynamic Programming is a mathematical technique to optimize a sequence of interrelated decisions, providing a systematic procedure for determining the optimal combination of these decisions. The objective of Dynamic Programming is to optimize sequential decision-making processes. It has been applied in a multitude of fields and industries, ranging from economics to engineering. From the mathematical point of view, Dynamic Programming can be applied to linear or nonlinear problems involving either real or integer variables. The only applicability condition consists in the mathematical separability of the objective and constraint functions with respect to the different decision variables. Dynamic Programming can tackle processes either deterministic or stochastic in nature, with a continuous or a discrete stage evolution, with both finite and infinite problem duration. Currently, the field of application of dynamic programming has become even more diverse, targeting optimization problems that can be reformulated as multi-stage decision processes. Some of the main areas of decision making such as Artificial Intelligence, Automatic control and Operations Research, make use of the paradign of Dynamic Programming.

B. *Fuzzy Dynamic Programming*

Dynamic programming was one of the earliest fundamental methodologies to which fuzzy sets was applied [Bellman and Zadeh, [3], leading to what is presently called *fuzzy dynamic programming* [5], [6]. Fuzzy dynamic programming has been applied successfully to multi stage decision making problems in a multitude of areas, with real world applications like civil and environmental engineering (integrated regional development, water resources operation and design, pollution control modeling), transportation (traffic planning and routing), energetic systems, health care, control systems, aerospace systems, etc. A significant body of work emerged since dynamic programming started being applied in conjunction with fuzzy representation.

Formally, let *X* be a space of options, then, given a fuzzy goal *G* in *X* characterized by the fuzzy membership function $\mu_G(x)$ and a fuzzy constraint *C* in *X* characterized by the fuzzy membership function $\mu_C(x)$, a fuzzy decision *D* in *X* which satisfies *C* while achieving *G* will have a fuzzy membership function $\mu_D(x)$ defined by:

$$\mu_D(x)=d(\mu_C(x),\mu_G(x))\in[0,1],\forall x\in X \qquad (22)$$

which provides for each $x\in X$ a measure of performance ranging from 1 for an excellent feasible decision to 0 for a very bad or unfeasible decisions, with intermediate values. In [7] function *d* is the fuzzy *and* operator which can be taken such as:

$$\mu_C(x)\wedge\mu_G(x)=\min(\mu_C(x),\mu_G(x)) \quad \forall x\in X \qquad (23)$$

In that case, the optimal decision with respect to $x\in X$ will be such that:

$$\mu_D(x^*)=\sup_{x\in X}(\mu_C(x)\wedge\mu_G(x)) \qquad (24)$$

Another common realization of the fuzzy *and* operator is:

$$\mu_C(x)\wedge\mu_G(x)=\mu_C(x)\cdot\mu_G(x) \quad \forall x\in X \qquad (25)$$

It appears that in both cases, the constraint and the goal are treated with the same priority since:

$$\mu_C(x)\wedge\mu_G(x)=\mu_G(x)\wedge\mu_C(x) \qquad (26)$$

and

$$\mu_C(x)\cdot\mu_G(x)=\mu_G(x)\cdot\mu_C(x) \quad \forall x\in X \qquad (27)$$

However for many applications, feasibility is a condition to be considered prior to any assessment of the degree of achievement of the goal. What can be expected is that a more appropriate fuzzy *and* function $d(\mu_C,\mu_G)$ fill the following conditions:

- $d(0,\mu_G)=0$ and $d(\mu_C,0)=0$;
- *d* is increasing with respect to both arguments;
- *d* is not commutative.

Examples of candidate *d* functions are:

$$d(\mu_C,\mu_G)=\mu_C{}^{\alpha}\cdot\mu_G \quad \text{with} \quad \alpha>1 \qquad (28)$$

$$d(\mu_C,\mu_G)=\frac{e^{\mu_C}-1}{e-1}\cdot\mu_G \qquad (29)$$

$$d(\mu_C,\mu_G)=\begin{cases}0 \;\; if\; \mu_C<s_{\min}\\ \mu_C\cdot\mu_G \quad if\; s_{\min}\le\mu_C\le\cdot s_{\max}\\ \mu_G\; if\; \mu_C>s_{\max}\end{cases} \qquad (30)$$

with $\quad 0\le s_{\min}<s_{\max}\le 1$

Here it will be considered that an analogous reasoning stands for the case of multiple fuzzy constraints and fuzzy goals, even if defined in different spaces.

Suppose that the fuzzy constraint C is defined on a fuzzy set in $X=\{x\}$, the fuzzy goal G is defined on a fuzzy set $Y=\{y\}$, and a function $f: X \rightarrow Y, y = f(x)$ is known. Typically, X and Y are decisions sets and their outcomes, respectively. Now the induced fuzzy goal G' in X generated by G in Y is given by:

$$\mu_{G'}(x) = \mu_G(f(x)), \text{ for each } x \in X \qquad (31)$$

with both G' and C being defined as fuzzy sets in the same space X. The min-type fuzzy decision is

$$\mu_D(x) = \mu_C(x) \hat{\wedge} \mu_{G'}(x) = \mu_C(x) \hat{\wedge} \mu_G(f(x)) \text{ for each } x \in X \qquad (32)$$

Then, for n fuzzy constraints defined in X, $C_1, \ldots, C_n$, m fuzzy goals defined in Y, $G_1, \ldots, G_m$, and a function $y = f(x)$, then the *min-type fuzzy decision* is given by:

$$\mu_D(x) = (\mu_{C_1}(x) \wedge \cdots \wedge \mu_{C_m}(x)) \hat{\wedge} (\mu_{G_1}(f(x)) \wedge \ldots \wedge \mu_{G_n}(f(x))) \quad \forall x \in X \qquad (33)$$

C. Dual Fuzzy Dynamic Programming

In this case, we consider the following problem:

$$\max \sum_{n=1}^{N-1} g_n(s_n, x_n) \qquad (34)$$

with $$g_n(s_n, x_n) = c_n(s_n, x_n) + \varepsilon \cdot d_n(s_n, x_n) \qquad (35)$$

where c_n and d_n are real valued functions and the feasibility conditions are supposed to be given by:

$$s_{n+1} = T(s_n, x_n) \in S \text{ and } x_n \in X_{S_n}, s_1 \text{ given} \qquad (36)$$

The adopted mathematical formalism, already proposed in [8] avoids some of the numerical difficulties appointed in the previous section.

Here N is the horizon of optimization, T represents the transition of the process from state s_n when decision x_n is taken, the resulting state is s_{n+1}. X_{S_n} is the set of feasible decisions according to current state s_n of the process.

A transition graph $G = [S, X]$ is built from the initial state s_1 by considering all feasible decisions from each state of each stage to the next stage:

$$S = \bigcup_{n=0}^{N-1} T^n(s_1) \text{ and } X = \bigcup_{n=0}^{N-1} \left(T^n(s_1), X_{S_n}\right) \qquad (37)$$

The optimality principle of dynamic programming can be put into action here to built from a stage to the next an optimal solution tree once fuzzy dual performances can always be compared according to (6), (7), (8) and (9) . For that, the fuzzy dual comparison proposed in paragraph II.B is used. When the performance of a path to a state is considered superior to any other path to this state with a degree of certainty c higher than 0.6, this path with the corresponding decision to reach it from the previous stage is retained. While, when $0.4 \leq c \leq 0.6$, the two fuzzy dual performances are considered very close and any of them can be taken as superior.

Then supposing that Γ_{nj} is the set of states of stage n-1 from which it is possible to reach state j of stage n, the retained decision from stage n-1 to state j of stage n will be associated to a state k_n^* of stage n-1 such as:

$$k_n^* = \arg\max_{k \in \Gamma_{n-1j}} \left\{ G_{n-1}^k + g_n(k,(k,j)) \right\} \qquad (38)$$

where

$$G_{n-1}^k = \sum_{m=1}^{n-1} g_m(k_{m-1}^*, (k_{m-1}^*, k_m^*)) \qquad (39)$$

and where a resulting degree certainty is given by:

$$c_n^j = \min_{k \in \Gamma_{n-1,j}, k \neq k_n^*} c_{n,k,j} \qquad (40)$$

where $c_{n,k,j}$ is attached to the degree of certainty of the fuzzy dual comparison of $G_{n-1}^k + g_n(k,(k,j))$ with $G_{n-1}^{k_n^*} + g_n(k_n^*, (k_n^*, j))$.

Then to each state j of each stage n is attached:

- a fuzzy dual performance given by $G_{n-1}^{k_n^*} + g_n(k_n^*, (k_n^*, j))$ and representing the deterministic aspects (the real part of the performance index) as well as the degree of uncertainty (the dual part of the performance index)
- a degree of certainty c_n^j of having chosen the best solution to reach state j at stage n.

The optimal sequence of decisions will follow from one stage to the next, the path from the initial state at the initial stage to a best performance state at the final stage.

V. CASE STUDY

A. Airport planning

Airport planning is in general a long term planning issue which has at its core the following objectives [9]: optimized infrastructure development costs and functionality, optimized economic and operational performance and a high degree of flexibility in order to integrate all the shifts in demand and potential disturbances according to the airport future needs and level of growth. The new business culture concepts that airports need to embrace includes strong air service competitor advantages, capability of taking long-term risks, adopting the stakeholder collaborative decision making culture, diversifying the revenues sources and most of all putting the passenger at the core of the business.

The construction of a new airport or the extension of an existing one requires significant investments and many times public-private partnerships have been considered in order to make feasible such projects [10]. One characteristic of these projects is uncertainty with respect to financial and environmental impacts on the medium to long term. Another one is the multistage nature of these types of projects. While many airport development projects have been a success like Munich Airport or Palma de Mallorca Airport, some others have turned into a nightmare for their promoters. For the illustration of the approach, the case of a national airport expected to gain an international position has been considered. The airport is supposed to be managed under a BOT agreement (Build – Operate – Transfer) over a future period of twenty five years. The financial risk of the concessionaire is to be unable to recover its investment, operating and maintenance expenses in the project. In this type of situation, the project proponent is facing a significant amount of risk that needs to be assessed and mitigated.

B. Costs, Revenues and Decisions

Different traffic types leading to costs and revenues can be considered in airports, they cover passengers and freight flows as well as aircraft traffic which is related with the level of these flows. Let the level of predicted potential demand for traffic type i along the planning horizon K be given by $D_k^i, i \in I, k \in \{1, 2, \cdots, K\}$, where I is the set of traffic activities. The necessary aircraft traffic T_k^i to cope with a predicted passenger demand level D_k^i can be approximated by:

$$T_k^i = D_k^i / (S_k^i \alpha_k^i) \tag{41}$$

where S_k^i is the mean capacity of aircraft type i at time k corrected by the expected mean load factor α_k^i. The rate of return, r_k^i, associated with the traffic of type i at time k, depends on the investments made until that period. Let the potential airport passenger processing capacity be $C_k^{P_i}$ and the potential aircraft movements processing capacity be C_k^{Ti}, then the estimated level of demand of type i at period k, $\overline{D}_k^i$, is such as:

$$\overline{D}_k^i = \min\{D_k^i, C_k^{Pi}, S_k^i C_k^{T_i}\} \tag{42}$$

Let L_i be the number of candidate upgrades, which can be performed for traffic, type i at the considered airport. Let θ_l^i be the period (an integer) at which upgrade l for traffic type i is planned to be done. When a project is retained, the corresponding value of θ_l^i is within the set $\{1,2,..., K\}$ and when it is not retained $\theta_l^i = K+1$, $l \in \{1,2,..., L_i\}$.

Different types of constraints may be found between interrelated projects:

- Sequential constraints: Technical considerations impose in general sequential constraints, so it is supposed that for given a type of traffic i and a pair of projects (l, l'), there may be constraints such as:

$$\exists l, l' \in \{1, \cdots, L_i - 1\}, i \in I : \theta_l^i \le \theta_{l'}^i \tag{43}$$

- Exclusion constraints such as if project l for traffic type i is retained, a set of concurrent or contradictory projects will be dismissed:

$$\theta_l^i \in \{1,2,..., K\} \Rightarrow \theta_{l'}^i = K+1, l' \in \Lambda_l^i \subset \{1, \cdots, L_i\} \tag{44}$$

- Inclusion constraints such as if project l for traffic type i is retained, a set of complementary projects related with other traffic should be performed altogether:

$$\theta_l^i \in \{1,2,..., K\} \Rightarrow \theta_{l'}^j = \theta_l^i, \; l' \in M_l^i \subset \{1, \cdots, L_j\} \tag{45}$$

Since the different types of traffic may use common resources in the airport, global capacity constraints must be satisfied. Let Δ_k be the set of projects which have been retained until period k, then the corresponding capacities with respect to passengers and flights are $C_k^{Pi}(\Delta_k)$ and $C_k^{T_i}(\Delta_k)$.

Let $c_l^{ik}(\Delta_k)$ be the cost of upgrade l with respect to traffic type i when performed at period k. Revenues R_k^i from traffic type i at period k will be supposed to be given by:

$$R_k^i = r_k^i \cdot \overline{D}_k^i(\Delta_k) \tag{46}$$

where r_k^i is the corresponding service rates.

C. Fuzzy Dual Performance Assessment

Let the fuzzy dual representations of the effective levels of respectively rates of net return, demands and upgrade costs be given by:

$$r_k^i = r_k^{iL} + \varepsilon \, r_k^{iD} \tag{47}$$

$$\overline{D}_k^i(\Delta_k) = \overline{D}_k^{iL}(\Delta_k) + \varepsilon \, \overline{D}_k^{iD}(\Delta_k) \tag{48}$$

$$c_l^{ik}(\Delta_k) = c_l^{ikL}(\Delta_k) + \varepsilon \, c_l^{ikD}(\Delta_k) \tag{49}$$

where the likely components are indexed by L and the dual components are indexed by D. In many situations, the likely components can be associated with mean estimated values while the dual components can be associated with their corresponding standard deviations.

The expression of the fuzzy dual net present value is given by:

$$\pi([\theta_l^i], l \in \{1, \cdots, L_i\}, i \in I) = \pi^L + \varepsilon \, \pi^D \tag{50}$$

where

$$\pi^L([\theta_l^i], l, i \in I) = \sum_{i \in I} \left(\sum_{k=1}^{K} \left(\frac{r_k^{iL}}{(1+\rho)^k} \overline{D}_k^{iL}(\Delta_k) \right) - \sum_{\substack{l=1 \\ \theta_l^i \le K}}^{L_i} \left(\frac{c_l^{ikL}(\Delta_k)}{(1+\rho)^{\theta_\rho^i}} \right) \right) \tag{51}$$

and

$$\pi^D([\theta_l^i], l \in \{1, \cdots, L_i\}, i \in I) = \sum_{i \in I} \left(\begin{array}{l} \sum_{k=1}^{K} \left(\frac{r_k^{iL} \cdot \overline{D}_k^{iD}(\Delta_k) + r_k^{iD} \cdot \overline{D}_k^{iL}(\Delta_k)}{(1+\rho)^k} \right) \\ - \sum_{\substack{l=1 \\ \theta_l^i \le K}}^{L_i} \left(\frac{c_l^{ikD}(\Delta_k)}{(1+\rho)^{\theta_\rho^i}} \right) \end{array} \right) \tag{52}$$

D. Application

Figure 4 displays the dynamic programming decision graph associated to an airport plan development including two new runways and two terminal buildings over a period of 25 years divided in five stages of five years duration and corresponding to five different operational configurations for the airport. Here 31 different paths lead to the states of the final stage while 20 different states at equal or different stages must be evaluated following relations (50), (51) and (52).

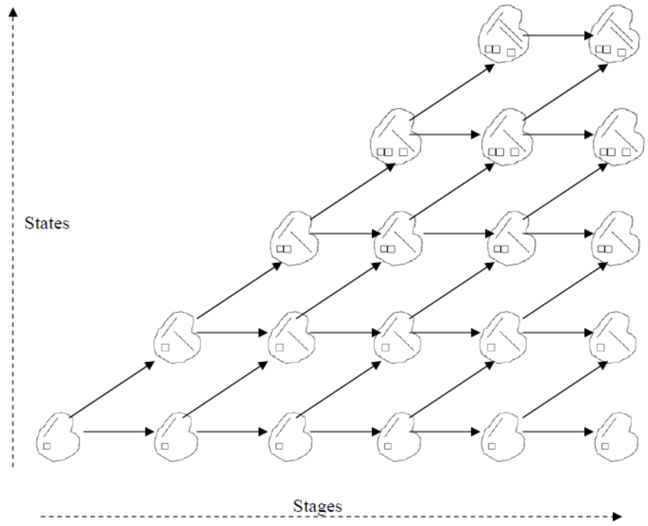

Fig. 4 Dynamic Programming Decision Graph

VI. CONCLUSION

This communication has considered sequential optimization problems where uncertainty is represented through fuzzy dual numbers. This formalism allows to limit the problem complexity and leads to the proposal of the Fuzzy Dynamic Programming approach which is introduced as a special case of Fuzzy Dynamic Programming. There the computational burden is turned feasible even when a new performance index, the degree of certainty, must be computed for each states in the sequential search process. The result of this approach is to provide to the final decider a map of the decision space (in fact a decision graph, see figure 4). This map informs for each state at each stage about intervals for expected returns and about the degree of certainty to have chosen the right sequence of decision leading to this state. A case study considering risk analysis in a long term airport planning situation has been developed.

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