Introduction to Fuzzy Dual Mathematical Programming

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Abstract. In this communication the formulation of optimization problems using fuzzy dual parameters and variables is introduced to cope with parametric or implementation uncertainties. It is shown that fuzzy dual programming problems generate finite sets of deterministic optimization problems, allowing to assess the range of the solutions and of the resulting performance at an acceptable computational effort.

Keywords. fuzzy dual numbers, fuzzy dual calculus, optimization, mathematical programming

Introduction

In general optimization problems assume implicitly that their parameters (cost coefficients, limit values for decision variables, boundary levels for constraints) are perfectly known while very often for real problems this is not exactly the case [1]. Different approaches have been proposed in the literature to cope with this difficulty. A first approach has been to perform around the nominal optimal solution numerical post optimization sensibility analysis [2]. When some probabilistic information about the values of the uncertain parameters is available, stochastic optimization techniques [3] may provide the most expected optimal solution. When these parameters are only known to remain within some intervals, robust optimization techniques [4] have been developed to provide robust solutions. The fuzzy formalism has been also considered in this case as an intermediate approach to represent the parameter uncertainties and provide fuzzy solutions [5]. These different approaches result in general into a very large amount of computation which turns them practically unfeasible.

In this communication, a new formalism based on fuzzy dual numbers is proposed to diminish the computational burden when dealing with uncertainty in mathematical programming problems.

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The adopted formalism considers fuzzy dual numbers which have been introduced recently by two of the authors [6] and which can be seen as a simplified version of fuzzy numbers adopting some elements of classical dual number calculus [7] and [8]. Indeed, the proposed special class of numbers, dual fuzzy numbers, integrates the nilpotent operator ε of dual numbers theory while considering symmetrical fuzzy numbers. Then uncertain values are characterized by only three parameters: a mean value, an uncertainty interval and a shape parameter.

In this communication, first are introduced the elements of fuzzy dual calculus useful to tackle the proposed issue: basic operations as well as strong fuzzy dual and weak fuzzy dual partial orders and fuzzy dual equality. Then two classes of fuzzy dual mathematical programming problems are considered: those where uncertainty relays only in the parameters of the problem and those for which the implementation of the solution is subject to uncertainty. In both situations, the proposed formalism is developed and used to identify the expected performance of the solutions.

1. Fuzzy Dual Numbers

The set of fuzzy dual numbers is the set $\tilde{\Delta}$ of numbers of the form $u = a + \varepsilon b$ such as $a \in R$, $b \in R^+$ where $r(u) = a$ is the primal part and $d(u) = b$ is the dual part of the fuzzy dual number.

A crisp fuzzy dual number will be such as b is equal to zero, losing its fuzzy dual attribute. To each fuzzy dual number $a + \varepsilon b$ is attached a fuzzy symmetrical number whose membership function μ is such that:

$$\mu(x) = \begin{cases} 0 \; if \;\; x \leq a - b \;\; or \; x \geq a + b \\ \quad \mu(x) = \mu(2a - x) \\ \quad x \in [a - b, a + b] \end{cases} \tag{1}$$

where μ is an increasing function between a-b and a with $\mu(a)=1$.

1.1. Operations with Fuzzy Dual Numbers

Different basic operations can be defined on $\tilde{\Delta}$ [9]. First, the fuzzy dual addition $\tilde{+}$, is given by:

$$(x_1 + \varepsilon\, y_1)\,\tilde{+}\,(x_2 + \varepsilon\, y_2) = (x_1 + x_2) + \varepsilon\,(y_1 + y_2) \tag{2}$$

where the neutral element of the fuzzy dual addition is $(0 + 0\varepsilon)$, written $\tilde{0}$.

Then the fuzzy dual product, written $\tilde{\bullet}$, is given by:

$$(x_1 + \varepsilon\, y_1)\,\tilde{\bullet}\,(x_2 + \varepsilon\, y_2) = (x_1 . x_2 + \varepsilon(|x_1| \cdot y_2 + |x_2| \cdot y_1)) \tag{3}$$

The fuzzy dual product has been chosen here in a way to preserve the fuzzy interpretation of the dual part of the fuzzy dual numbers, so it is different of the product of dual calculus. The neutral element of fuzzy dual multiplication is $(1+0\varepsilon)$, written $\tilde{1}$. It is easy to check that internal operations such as fuzzy dual addition and fuzzy dual multiplication are commutative and associative. The fuzzy dual multiplication is distributive with respect to the fuzzy dual addition since operator ε is according to Eq. (3) such as:

$$\varepsilon \,\tilde{\bullet}\, \varepsilon = \tilde{0} \tag{4}$$

Comparing with common fuzzy calculus, fuzzy dual calculus appears to be much less demanding in computer resource [10] and [11].

1.2. Fuzzy Dual Vectors

Let E be an Euclidean space of dimension p over R then we define the set of fuzzy dual vectors $\tilde{E}$ as the pairs of vectors which are taken from the Cartesian product $E \times E^{+}$, where E^{+} is the positive half-space of E. Basic operations can be defined over $\tilde{E}$:
Addition:

$$(a+\varepsilon\, b)+(c+\varepsilon\, d)=(a+c)+\varepsilon(b+d) \qquad \forall a,c \in E, \forall b,d \in E^{+} \tag{5}$$

Multiplication by a fuzzy dual scalar $\lambda + \varepsilon\, \mu$:

$$(\lambda+\varepsilon\, \mu)(a+\varepsilon\, b)=\lambda\, a+\varepsilon(|\lambda|\, b+\mu|a|) \qquad \forall \lambda+\varepsilon\, \mu \in \tilde{\Delta}, \forall a+\varepsilon\, b \in \tilde{E} \tag{6}$$

A pseudo scalar product is defined by:

$$u * v = r(u).r(v)+\varepsilon\,(|r(u)|.d(v)+d(u).|r(v)|) \quad \forall\, u,v \in \tilde{E} \tag{7}$$

where "*" represents the inner product in $\tilde{E}$ and "." represents the inner product in E.

2. Fuzzy Dual Inequalities

With the objective to make possible the comparison of fuzzy dual numbers as well as the identification of extremum values between fuzzy dual numbers, a new operator from $\tilde{\Delta}$ to R^{+}, called *fuzzy dual pseudo norm*, is introduced.

2.1. Fuzzy dual pseudo norm

Let us introduce the proposed operator:

$$\forall a+\varepsilon\, B \in \tilde{\Delta} : \|a+\varepsilon\, b\|_{D} = |a| + \rho\, b \in R^{+} \tag{8}$$

where ρ is a shape parameter associated with the considered fuzzy dual number which is given by:

$$\rho = \frac{1}{2b} \int_{x \in R} \mu(x) dx \in [0, 1] \tag{9}$$

In the case of fuzzy dual numbers with symmetrical triangular membership functions, ρ = ½ while for crisp fuzzy dual numbers, $\rho = 0$. In this paper it is supposed that the considered fuzzy dual numbers have the same shape, i.e. a common ρ value.

It is straightforward to establish that the operator defined in Eq.(8), whatever the value of the shape parameter, satisfies the characteristic properties of a norm:

$$\forall\ a + \varepsilon\, b \in \tilde{\Delta} : \|a + \varepsilon\, b\| \geq 0 \tag{10}$$

$$\forall a \in R, \forall b \in R^+ \qquad \|a + \varepsilon\ b\| = 0 \Rightarrow a = b = 0 \tag{11}$$

$$\|(a + \varepsilon\, b) + (\alpha + \varepsilon\, \beta)\| \leq \|a + \varepsilon\, b)\| + \|\alpha + \varepsilon\, \beta\| \qquad \forall a, \alpha \in R, \forall b, \beta \in R^+ \tag{12}$$

$$\|\lambda(a + \varepsilon\, b)\|_D = |\lambda| \|a + \varepsilon\, b\|_D\ \forall a, \lambda \in R, \forall b \in R^+ \tag{13}$$

However, since the set of dual numbers $\tilde{\Delta}$ is not a vector space, the proposed operator can be only regarded as a pseudo norm.

The fuzzy dual pseudo norm of a fuzzy dual vector u can be introduced as (here $\|\ \|$ is the Euclidean norm associated to E):

$$\|u\|_D = \|r(u)\| + \rho\, \|d(u)\| \tag{14}$$

2.2. Strong and weak fuzzy dual inequalities

Partial orders between fuzzy dual numbers can be introduced using this pseudo norm. Depending if fuzzy dual numbers overlap or not, strong and weak partial orders can be introduced.

A *strong* fuzzy dual partial order written $\breve{\geq}$ is defined over $\tilde{\Delta}$ by:

$$\begin{aligned} &\forall\ a_1 + \varepsilon\, b_1\ , a_2 + \varepsilon\, b_2 \in \tilde{\Delta} : a_1 + \varepsilon\, b_1 \breve{\geq} a_2 + \varepsilon\, b_2 \\ &\Leftrightarrow a_1 - \rho\, b_1 \geq a_2 + \rho\, b_2 \end{aligned} \tag{15}$$

In that case there is no overlap between the membership functions associated with the two fuzzy dual numbers and the first one is definitely larger than the second one.

A *weak* fuzzy dual partial order written $\hat{\geq}$ is defined over $\tilde{\Delta}$ by:

$$\begin{aligned} &\forall a_1 + \varepsilon b_1, a_2 + \varepsilon b_2 \in \tilde{\Delta} : a_1 + \varepsilon b_1 \hat{\geq} a_2 + \varepsilon b_2 \\ &\Leftrightarrow a_1 + \rho b_1 \geq a_2 + \rho b_2 \geq\ a_1 - \rho b_1 \geq a_2 - \rho b_2 \end{aligned} \tag{16}$$

In that case there is an overlap between the membership functions associated with the two fuzzy dual numbers and the first one appears to be partially larger than the second one.

A fuzzy dual equality, written $\cong$, can be defined between two fuzzy dual numbers by:

$$\begin{gathered}\forall\ a_1 + \varepsilon\, b_1\ , a_2 + \varepsilon\, b_2 \in \tilde{\Delta} : \|a_1 + \varepsilon\, b_1\| \cong \|(a_2 + \varepsilon\, b_2)\| \\ \Leftrightarrow a_2 \in [a_1 - \rho\, b_1, a_1 + \rho\, b_1]\ et\ a_1 \in [a_2 - \rho\, b_2, a_2 + \rho\, b_2]\end{gathered} \tag{17-a}$$

$$\begin{aligned}&\forall a_1 + \varepsilon b_1, a_2 + \varepsilon b_2 \in \tilde{\Delta} : a_1 + \varepsilon b_1 \cong a_2 + \varepsilon b_2 \\ &\Leftrightarrow a_1 + \rho b_1 \geq a_2 + \rho b_2 \geq\ a_2 - \rho b_2 \geq a_1 - \rho b_1 \\ &or\ a_2 + \rho b_2 \geq a_1 + \rho b_1 \geq\ a_1 - \rho b_1 \geq a_2 - \rho b_2\end{aligned} \tag{17-b}$$

In this last case there is a complete overlap of the membership functions associated with the two fuzzy dual numbers.

Then when considering two fuzzy dual numbers, they will be in one of the above situations (no overlap, partial overlap or full overlap): strong fuzzy dual inequality, weak fuzzy dual inequality or fuzzy dual equality.

2.3. Extremum operators

The *max* and the *min* operators over two or more fuzzy dual numbers can now be defined. Let $c+\varepsilon\,\gamma$ be the fuzzy dual maximum of fuzzy dual numbers $a + \varepsilon\,\alpha$ and $b+ \varepsilon\,\beta$:

$$c + \varepsilon \cdot \gamma = \max\{a + \varepsilon \cdot \alpha, b + \varepsilon \cdot \beta\} \tag{18}$$

then:

$$c = \max\{a, b\} \tag{19.a}$$

$$\gamma = \max\{a + \rho\,\alpha, b + \rho\,\beta\} - \max\{a, b\} \tag{19.b}$$

Let d+ ε δ be the fuzzy dual minimum of fuzzy dual numbers $a + \varepsilon\,\alpha$ and $b+ \varepsilon\,\beta$:

$$d + \varepsilon \cdot \delta = \min\{a + \varepsilon \cdot \alpha, b + \varepsilon \cdot \beta\} \tag{20}$$

then:

$$d = \min\{a, b\} \tag{21.a}$$

$$\delta = \min\{a + \rho\,\alpha, b + \rho\,\beta\} - \min\{a, b\} \tag{21.b}$$

Observe that here the *max* and *min* operators produce new fuzzy dual numbers.

3. Mathematical Programming with Fuzzy Dual Parameters

Here is introduced the fuzzy dual formulation of uncertain mathematical programming problems.

3.1. Discussion

To illustrate the proposed approach the case of a linear programming problem with real variables where all parameters are uncertain and described by fuzzy dual numbers, is considered. The proposed approach can be easily extended to integer mathematical programming problems or to nonlinear mathematical programming problems, or to problems with different types of level constraints.

Let then define formally problem $\tilde{L}$ given by:

$$\min_{\underline{x}\in R^{n+}} \left| \sum_{i=1}^{n} \tilde{c}_i \, x_i \right| \tag{22}$$

under the constraints:

$$\sum_{i=1}^{n} \tilde{a}_{ki} \, x_i \geq \tilde{b}_k \quad k \in \{1,\cdots,m\} \tag{23}$$

and

$$x_i \in R^+ \quad i \in \{1,\cdots,n\} \tag{24}$$

where the coefficients $\tilde{a}_{hi}, \tilde{b}_k, \tilde{c}_i$ are uncertain parameters.

When the problem is a constrained cost minimization problem, the cost parameters $\tilde{c}_i$, although uncertain, remains positive and the absolute operator can be retrieved from expression of Eq. (22). Here is adopted the fuzzy dual hypothesis for the cost coefficients c_i, the technical parameters a_{ki} and the constraint levels b_k. This opens different perspectives to be considered when dealing with the parameter uncertainty. Here are considered three different cases:

- the nominal case (a standard deterministic linear programming problem) in which the dual parts of the parameters are zero;
- the pessimistic case where uncertainty adds to the cost and where constraints are strong ones,
- the optimistic case where uncertainty subtracts from the cost and the constraints are weak ones.

The nominal case corresponds to a standard mathematical programming problem. The analysis of the pessimistic case is developed here with more detail and can be transposed easily to the study of the optimistic case.

3.2. Minimum Performance Bound

In the pessimistic case, problem L_+ is formulated which is a fuzzy dual linear programming problem with fuzzy dual constraints and real decision variables and is written as:

$$\min_{\underline{x}\in R^{n+}} \left\| \sum_{i=1}^{n} (c_i + \varepsilon\, d_i)\, x_i \right\| \tag{25}$$

under strong inequality constraints:

$$\sum_{i=1}^{n} (a_{ki} + \varepsilon\, \alpha_{ki})\, x_i \;\breve{\geq}\; b_k + \varepsilon\beta_k \quad k \in \{1, \cdots, m\} \tag{26}$$

$$\text{and} \qquad x_i \in R^{+} \quad i \in \{1, \cdots, n\} \tag{27}$$

where $c_i, d_i, a_{ki}, \alpha_{ki}, b_k, \beta_k$ are given.

This problem corresponds to the minimization of the worst estimate of total cost with satisfaction of strong level constraints. Here variables x_i are supposed to take real positive values, but they could take also fully real or integer values. In the case in which the d_i are zero, the uncertainty is relative to the feasible set. Problem L_+ is equivalent to the following problem in R^{+n}:

$$\min_{\underline{x}\in R^{n+}} \left| \sum_{i=1}^{n} c_i x_i \right| + \rho \sum_{i=1}^{n} d_i x_i \tag{28}$$

under the hard constraints:

$$\sum_{i=1}^{n} (a_{ki} - \rho\alpha_{ki})\, x_i \geq b_k + \rho\, \beta_k \quad k \in \{1, \cdots, m\} \tag{29}$$

and

$$x_i \geq 0 \quad i \in \{1, \cdots, n\} \tag{30}$$

It appears that the proposed formulation leads to minimize a combination of the values of the nominal criterion and of its degree of uncertainty. In the case in which the cost coefficients are positive this problem reduces to a classical linear programming problem over R^{+n}. In the general case, since the quantity $\sum_{i=1}^{n} c_i x_i$ will have at solution a particular sign, the solution $\underline{x}^{+}$ of problem L_+ will be the one corresponding to the minimum of:

$$\{ \ \min_{\underline{x}\in R^{n+}} \left(\sum_{i=1}^{n} c_i \bar{x}_i + \rho \sum_{i=1}^{n} d_i \bar{x}_i \right) \ , \min_{\underline{x}\in R^{n+}} \left(\sum_{i=1}^{n} c_i \vec{x}_i + \rho \sum_{i=1}^{n} d_i \vec{x}_i \right) \ \} \tag{31}$$

where $\underline{\bar{x}}$ is solution of problem:

$$\min_{\underline{x}\in R^{n+}} \Big(\sum_{i=1}^{n} c_i x_i + \rho \sum_{i=1}^{n} d_i x_i \Big) \tag{32}$$

under the hard constraints:

$$\sum_{i=1}^{n} (a_{ki} - \rho\ \alpha_{ki})\, x_i \geq b_k + \rho\, \beta_k \quad k \in \{1, \cdots, m\} \tag{33}$$

$$\sum_{i=1}^{n} c_i x_i \geq 0 \quad \text{and} \quad x_i \geq 0 \quad i \in \{1, \cdots, n\} \tag{34}$$

and where $\underline{\vec{x}}$ is solution of problem:

$$\min_{\underline{x} \in R^{n+}} (\rho \sum_{i=1}^{n} d_i x_i - \sum_{i=1}^{n} c_i x_i) \tag{35}$$

under the following constraints:

$$\sum_{i=1}^{n} (a_{ki} - \rho\,\alpha_{ki})\, x_i \geq b_k + \rho\,\beta_k \quad k \in \{1, \cdots, m\} \tag{36}$$

$$\sum_{i=1}^{n} c_i x_i \leq 0 \quad \text{and} \quad x_i \geq 0 \quad i \in \{1, \cdots, n\} \tag{37}$$

The fuzzy dual optimal performance of this program is then given by:

$$\sum_{i=1}^{n} (c_i + \varepsilon\, d_i)\, x_i^+ = \sum_{i=1}^{n} c_i x_i^+ + \varepsilon \sum_{i=1}^{n} d_i x_i^+ \tag{38}$$

Problems of Eqs. (32), (33) and (34) and of Eqs. (35), (36) and (37) are classical continuous linear programming problems which can be solved in acceptable time even for large size problems.

4.3 Performance analysis

It is of interest to consider the complementary problem L_- given by:

$$\min_{\underline{x} \in R^{n+}} \left| \sum_{i=1}^{n} c_i x_i \right| - \rho \sum_{i=1}^{n} d_i x_i \tag{39}$$

under the constraints:

$$\sum_{i=1}^{n} (a_{ki} + \rho\,\alpha_{ki})\, x_i \geq b_k - \rho\,\beta_k \quad k \in \{1, \cdots, m\} \tag{40}$$

and

$$x_i \geq 0 \quad i \in \{1, \cdots, n\} \tag{41}$$

and the nominal problem L_0 given by:

$$\min_{\underline{x} \in R^{n+}} \left| \sum_{i=1}^{n} c_i x_i \right|_i \tag{42}$$

under the nominal constraints:

$$\sum_{i=1}^{n} a_{ki} x_i \geq b_k \quad k \in \{1, \cdots, m\} \tag{43}$$

and

$$x_i \geq 0 \quad i \in \{1, \cdots, n\} \tag{44}$$

Let $\underline{x}^{-}$ and $\underline{x}^{0}$ be the respective solutions of problems of Eqs.(39), (40) and (41) and of Eqs. (42),(43) and (44), it will be instructive to compare in a first step the performances of problems L_+, L_- and L_0 where:

$$\left|\sum_{i=1}^{n} c_i x_i^{-}\right| - \rho\sum_{i=1}^{n} d_i x_i^{-} \leq \left|\sum_{i=1}^{n} c_i x_i^{0}\right| \leq \left|\sum_{i=1}^{n} c_i x_i^{+}\right| + \rho\sum_{i=1}^{n} d_i x_i^{+} \tag{45}$$

This allows to display the dispersion of results between the pessimistic view of problem L_+, the optimistic view of problem L_- and the neutral view of problem L_0.

Then in a second step, since $\underline{x}^{+}$ is feasible for problems L_- and L_0, it is of interest to compare the different performances when adopting the $\underline{x}^{+}$ solution:

$$\left|\sum_{i=1}^{n} c_i x_i^{+}\right| - \rho\sum_{i=1}^{n} d_i x_i^{+} \leq \left|\sum_{i=1}^{n} c_i x_i^{+}\right| \leq \left|\sum_{i=1}^{n} c_i x_i^{+}\right| + \rho\sum_{i=1}^{n} d_i x_i^{+} \tag{46}$$

to produce bounds to the effective performance of the solution.

4. Mathematical Programming with Fuzzy Dual Variables

Now we consider fuzzy dual programming problems with fuzzy dual parameters and decision variables as well. In that case problem V is formulated as:

$$\min_{x_i \in R, y_i \in R^{+}} \left\|\sum_{i=1}^{n} (c_i + \varepsilon\, d_i)(x_i + \varepsilon\, y_i)\right\| \tag{46}$$

under the strong constraints

$$\sum_{i=1}^{n} (a_{ki} + \varepsilon\, \alpha_{ki})\,(x_i + \varepsilon\, y_i) \,\breve{\geq}\, b_k + \varepsilon\beta_k \quad k \in \{1,\cdots,m\} \tag{48}$$

and

$$x_i \in R,\ y_i \geq 0 \quad i \in \{1,\cdots,n\} \tag{49}$$

The above problem corresponds to the minimization of the worst estimate of total cost with satisfaction of strong level constraints when there is some uncertainty not only on the values of the parameters but also on the ability to implement exactly what should be the optimal solution. According to Eq. (3), problem V can be rewritten as:

$$\min_{\underline{x} \in R^n, \underline{y} \in R^{n+}} \left\|\sum_{i=1}^{n} (c_i x_i + \varepsilon(|x_i| d_i + |c_i| y_{))i}\right\| \tag{50}$$

under constraints Eq. (41) and :

$$\sum_{i=1}^{n} (a_{ki}\, x_i + \varepsilon\,(\alpha_{ki}|x_i| + |a_{ki}| y_i)) \,\breve{\geq}\, b_k + \varepsilon\beta_k \quad k \in \{1,\cdots,m\} \tag{51}$$

which is equivalent in $R^n \times R^{n+}$ to the following mathematical programming problem:

$$\min_{\underline{x} \in R, \underline{y} \in R^{n+}} C(\underline{x}, \underline{y}) = \left|\sum_{i=1}^{n} c_i x_i\right| + \rho\sum_{i=1}^{n} (d_i|x_i| + |c_i|\, y_i) \tag{52}$$

under constraints of Eq. (41) and hard constraints:

$$\sum_{i=1}^{n} (a_{ki}\, x_i - \rho(\alpha_{ki}|x_i| + |a_{ki}|\, y_i)) \geq b_k + \rho\,\beta_k \quad k \in \{1,\cdots,m\} \tag{53}$$

Let $A(\underline{x},\underline{y})$ be the set defined by:

$$A(\underline{x},\underline{y}) = \left\{ \begin{array}{l} \underline{x} \in R^n, \underline{y} \in R^{+n} : \\ \sum_{i=1}^{n} (a_{ki}\ x_i - \rho(\ \alpha_{ki}|x_i| + |a_{ki}|\ y_i)) \geq b_k + \rho\ \beta_k \\ \quad k \in \{1, \cdots, m\} \end{array} \right\} \tag{54}$$

then

$$\forall x \in R^n, \forall y \in R^{n+} \qquad A(\underline{x},\underline{y}) \subset A(\underline{x},\underline{0}) \ \text{ and } \ C(\underline{x},\underline{y}) \geq C(\underline{x},\underline{0}) \tag{55}$$

It appears, as expected, that the case of no diversion of the nominal solution is always preferable. In the case in which the diversion from the nominal solution is fixed to $\bar{y}_i, i \in \{1, \cdots, n\}$, problem *V* has the same solution than problem *V'* given by:

$$\min_{\underline{x} \in R^n} \left| \sum_{i=1}^{n} c_i x_i \right| + \rho \sum_{i=1}^{n} d_i |x_i| \tag{56}$$

under constraints Eq. (41) and:

$$\sum_{i=1}^{n} (a_{ki}\ x_i - \rho\ \alpha_{ki}|x_i|) \geq b_k + \rho\,(\beta_k + \sum_{i=1}^{n} |a_{ki}|\ \bar{y}_i) \qquad k \in \{1, \cdots, m\} \tag{57}$$

The fuzzy dual optimal performance of problem of Eq. (46) will be given by:

$$\sum_{i=1}^{n} c_i\ x_i^* + \varepsilon \sum_{i=1}^{n} \ (|x_i^*| d_i + |c_i| y_i) \tag{58}$$

where $\underline{x}^*$ of problem *V'*.

Here also other linear constraints involving the other partial order relations over $\tilde{\Delta}$ (weak inequality and fuzzy equality) could be introduced in the formulation of problem *V* while the consideration of the integer version of problem *V* will lead also to solve families of classical integer linear programming problems.

The performance of the solution of problem *V* will be potentially diminished by the reduction of the feasible set defined by Eq. (54).

5. Conclusion

This study has considered mathematical programming problems presenting some uncertainty on the values of their parameters or in the implementation of the values for the decision variables. A special class of fuzzy numbers, fuzzy dual numbers, has been defined in such a way that the interpretation of their dual part as an uncertainty level remains valid through the basic operations on these numbers. A pseudo norm has been introduced, allowing the comparison between fuzzy dual expressions and leading to the definition of hard and weak constraints to characterize fuzzy dual feasible sets. Mathematical programming problems with uncertain parameters and variables have been considered under this formalism. The proposed solution approach leads to solve a finite collection of classical mathematical programming problems corresponding to nominal and extreme cases, allowing to the characterization of the expected optimal performance and solution. These results in a rather limited additional computational effort compared with classical approaches. The above approach could be easily extended to cope with fuzzy dual numbers of different shapes present in the same mathematical programming problem.

References

[1] M. Delgado, J. L. Verdegay and M. A. Vila, Imprecise costs in mathematical programming problems, *Control and Cybernetics*, **16**(1987), 114-121.

[2] T. Gal, H. J. Greenberg (Eds.), Advances in Sensitivity Analysis and Parametric Programming, Series: International Series in Operations Research & Management Science, Vol. 6, *Springer*, 1997.

[3] A. Ruszczynski and A. Shapiro. Stochastic Programming. Handbooks in Operations Research and Management Science, Vol. 10, *Elsevier*, 2003.

[4] A. Ben-Tal, L. El Ghaoui and A. Nemirovski, Robust Optimization. Princeton Series in Applied Mathematics, *Princeton University Press*, 2009.

[5] H. J. Zimmermann, Fuzzy Sets Theory and Mathematical Programming, in A. Jones et al. (eds.), Fuzzy Sets Theory and Applications, *D. Reidel Publishing Company*, 99-114, 1986.

[6] C. A. N. Cosenza and F. Mora-Camino, Nombres et ensembles duaux flous et applications, in French, Technical repport, Labfuzzy laboratory, COPPE/UFRJ, Rio de Janeiro, August 2011.

[7] W. Kosinsky, On Fuzzy Number Calculus, *International Journal of Applied Mathematics and Computer Science*, **16**(2006), 51-57.

[8] H. H. Cheng , Programming with Dual Numbers and its Application in Mechanism Design, *Journal of Engineering with Computers*, **10**(1994), 212-229.

[9] Mora_Camino F., O. Lengerke and C. A. N. Cosenza, *Fuzzy sets and dual numbers, an integrated approach*, Fuzzy sets and Knowledge Discovery Conference, Chongqing, China, 28-31 May 2012.

[10] H. Nasseri, Fuzzy Numbers: Positive and Nonnegative, *International Mathematical Forum*, **3**(2006), 1777-1780.

[11] E. Pennestrelli and R. Stefanelli, Linear Algebra and Numerical Algorithms using Dual Numbers, *Journal of Multibody Systems Dynamics*, **18**(2007), 323-344.