Invariant Unscented Kalman filtering : A Parametric formulation study for Attitude Estimation

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Abstract

The invariant unscented Kalman filtering (IUKF), relies on a geometrical-based constructive method for designing filters dedicated to nonlinear state estimation problems while preserving the physical invariances and systems symmetries. This can be achieved by using a geometrically adapted correction term based on an invariant output error. In this article, a special formulation of the attitude and heading estimation problem derives the invariant IUKF so that state and sigma-points are considered as a transformation group parameterization. The specific interest of this formulation is that only the invariant errors between the predicted state and the sigma-points must be known to determine the predicted outputs errors. As this is already computed during the prediction step, the computation complexity to find the covariance matrix of the invariant state estimation is greatly reduced.

Key words: Attitude estimation; Invariant filtering; Invariant Unscented Kalman filtering; Symmetries; Transformation group parametrization.

1 INTRODUCTION

Although dynamical systems possessing symmetries have been studied in control theory, few results taking benefit of system invariances for observers design exist today. Invariant nonlinear estimation theory appears as a young research domain whose first main contributions can be dated from the beginning of 2000s. At that time, several research works on nonlinear invariant observers have been led and provide a geometrical-based constructive method for designing observers able to estimate dynamical systems state vector while preserving their symmetries ([1–4]). Building upon both invariant frame and output-error, this peculiar kind of observers allows to formulate a state estimation error whose dynamics have a remarkable property: it does not depend on the followed trajectory. It requires however to tune an important number of setting parameters potentially when computing estimation gains, which can be cumbersome for complex system modeling. Thereafter, researchers have tried to develop more generic procedures which facilitate the design of invariant observers, by performing an automatic tuning of the correction gains which occurs in any filtering equation associated with nonlinear state observer.
Regarding the state of the art, there exist two major techniques : Invariant Extended Kalman Filter – IEKF or more recently the Invariant Unscented Kalman Filter IUKF and the Invariant Particle Filter – IPF.
The IEKF ([5–7, 15]) is characterized by a larger convergence domain, due to the exploitation of systems' symmetries within the estimation algorithm (*i.e.,* within filter equations and gains computation), and present very good performances in practice. An important well known drawback in this method is that it requires to linearize the system of differential equations which govern the invariant state estimation error dynamics. Such an operation appears suitable for simple system modeling but for more complex cases, this linearization may be difficult to carry out.
To overcome this problem, the UKF algorithm based on invariant framework has been recently proposed in ([14],[8–12]). It has been proved in these bibliographical references that an Invariant UKF-like estimator could be designed by using either a compatibility condition or a specific case in order to make equation more explicit for attitude estimation problem. In a similar spirit to research from a few years ago on the IEKF (Invariant Extended Kalman Filter) algorithm, the correction gains of this estimator, which are specifically designed to be invariant, may be deduced by performing the same computational steps as UKF-type filtering (either in factorized or non-factorized form). However, before we can integrate the procedure for computing the correction gains (an algorithm borrowed from unscented Kalman filtering) with invariant observer theory, a series of methodological developments are required, as described in this article. Similarly, an extension of nonlinear invariant observers has been made for Rao-Blackwellized Particle Filters (PF) that can be used for nonlinear state estimation ([13]). Invariant PFs (IPF) rely on the notion of *conditional invariance* which corresponds to classical system invariance properties, but once some state variables are assumed to be known. It is those known states that will be sampled throughout the estimation process. It is noteworthy that, for the obtained IPF, the Kalman gains computed are identical for all particles which drastically reduces the computational effort usually needed to implement any PF. Among methods described above, only a few tried to customize equations in order to make them

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more explicit for the special case of the Attitude and Heading Reference System (AHRS) ([14–16]). But none of them is able to reduce the computation cost. This paper focuses on a new formulation of the Invariant UKF-like estimator for AHRS in order to reduce this cost. The contributions of the paper include :

(1) The presentation in Section 2 of theoretical prerequisities dealing with unscented Kalman filtering where both invariant state and output error are introduced. The invariant framework dedicated to unscented Kalman filter is exceedingly convenient as filter equations can be specialized.

(2) The invariant unscented Kalman filter equations presented in Section 2 are applied on the system of differential equations that described the proposed IUKF in the benchmarking case of an attitude estimation system is derived. Our focus in Section 4 is on finding parametization group that reduce the computational cost of the filter.

Finally, the compuational results that appear in Theorem 2 are validated in Section 5 on the basis of numerical results. The performances reached by the UKF, the standard IUKF and the developed IUKF dedicated to AHRS named $\text{IUKF}_{\mathbf{x}}$ are compared. We provide preliminary results on the computational effort of algorithms validating the reduction of the computational complexity of the new IUKF parametric formulation.

2 Some preliminaries

This section introduces the unscented Kalman filter this article is concerned with, as well as the invariant unscented Kalman filter our results apply to.

2.1 Unsented Kalman Filter

The standard UKF framework ([17]) involves estimation of the state $\mathbf{x}_k \in \mathbb{R}^n$ of a discrete-time nonlinear dynamic system,

$$\begin{aligned} \mathbf{x}_{k+1} &= f(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{w}_k \\ \mathbf{y}_k &= h(\mathbf{x}_k, \mathbf{u}_k) + \mathbf{v}_k, \end{aligned} \tag{1}$$

where $\mathbf{y}_k \in \mathbb{R}^m$ is the output of the modeled system. $\mathbf{v}_k \in \mathbb{R}^m$ (resp. $\mathbf{w}_k \in \mathbb{R}^n$) refers to the discrete Gaussian process $\mathbf{w}_k \sim N(0, \mathbf{W}_k)$ (resp. observation $\mathbf{v}_k \sim N(0, \mathbf{V}_k)$). The UKF estimation process starts with the calculation of the scaled Unscented Transform (UT), in order to pick a minimal set of sample points, also called *sigma points*, around the mean state vector denoted by $\boldsymbol{\mathcal{X}}$, s.t. $\boldsymbol{\mathcal{X}}^{(0)}_{k|k} = \hat{\mathbf{x}}_{k|k}$. This calculation provides a set of $(2n+1)$ sigma points and also two series of $(2n+1)$ scalar weighting factors, denoted by $\{W^{(i)}_{(m)}\}$ and $\{W^{(i)}_{(c)}\}$ ($i \in [\![0\,; 2n]\!]$). These sigma points are then propagated through the nonlinear state $f(\cdot)$ and output $h(\cdot)$ equations, providing a cloud of evolving points. The mean $\hat{\mathbf{x}}_{k|k}$ and estimated covariance matrix $\mathbf{P}^{\mathbf{x}}_{k|k}$ of the transformed points are then computed based on their statistics. The mean and covariance of the initial state $\mathbf{x}_0$ are denoted $\hat{\mathbf{x}}_0$ and $\mathbf{P}^{\mathbf{x}}_0$, respectively. The unscented transform can be seen as a function (or functional) from $(f||h, \hat{\mathbf{x}}_k, \mathbf{P}^{\mathbf{x}}_k)$ to $(\hat{\mathbf{x}}_{k+1|k}||\hat{\mathbf{y}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k})$ depending if the unscented transform is applied on the process or (||) output equation (See appendix A):

$$(\hat{\mathbf{x}}_{k+1|k}||\hat{\mathbf{y}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k}||\tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k}) = \mathrm{UT}(f||h, \hat{\mathbf{x}}_k, \mathbf{P}^{\mathbf{x}}_k). \tag{2}$$

In terms of the unscented transform $\mathrm{UT}(\cdot)$ the *unscented Kalman filter* prediction and update steps can be written as follows :

- *Prediction:* Compute the predicted state mean $\hat{\mathbf{x}}_{k+1|k}$ and the predicted covariance $\mathbf{P}^{\mathbf{x}}_{k+1|k}$:

$$\begin{aligned} [\hat{\mathbf{x}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k}] &= \mathrm{UT}(f, \hat{\mathbf{x}}_{k|k}, \mathbf{P}^{\mathbf{x}}_{k|k}) \\ \mathbf{P}^{\mathbf{x}}_{k+1|k} &= \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k} + \mathbf{W}_k \end{aligned} \tag{3}$$

- *Update:* Compute the predicted mean $\hat{\mathbf{y}}_{k+1|k}$ and covariance of the measurement $\mathbf{P}^{\mathbf{y}}_{k+1|k}$, and the cross-corvariance of the state and measurement $\mathbf{P}^{\mathbf{xy}}_{k+1|k}$:

$$\begin{aligned} [\hat{\mathbf{y}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k}] &= \mathrm{UT}(h, \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}^{\mathbf{x}}_{k+1|k}) \\ \mathbf{P}^{\mathbf{y}}_{k+1|k} &= \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k} + \mathbf{V}_k \\ \mathbf{P}^{\mathbf{xy}}_{k+1|k} &= \sum_{i=0}^{2n} W^{(i)}_{(c)} (\hat{\mathbf{x}}^{(i)}_{k+1|k} - \hat{\mathbf{x}}_{k+1|k}) \\ &\times (\hat{\mathbf{y}}_{k+1|k} - \hat{\mathbf{y}}^{(i)}_{k+1|k}) \end{aligned} \tag{4}$$

  An estimation $\hat{\mathbf{x}}_{k+1|k+1}$ of $\mathbf{x}_{k+1}$ is then computed by the Kalman filtering equations :

$$\begin{aligned} \hat{\mathbf{x}}_{k+1|k+1} &= \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k}) \\ \mathbf{P}^{\mathbf{x}}_{k+1|k+1} &= \mathbf{P}^{\mathbf{x}}_{k+1|k} - \mathbf{K}_{k+1}\mathbf{P}^{\mathbf{y}}_{k+1|k}\mathbf{K}^T_{k+1}, \end{aligned} \tag{5}$$

  where $\mathbf{K}_{k+1} = \mathbf{P}^{\mathbf{xy}}_{k+1|k}{\mathbf{P}^{\mathbf{y}}_{k+1|k}}^{-1}$ and $\mathbf{y}_{k+1}$ is the raw measurements.

The linear correction $(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k})$ is weighted by the gain $\mathbf{K}_{k+1}$ in such a way as to minimize the covariance of the state estimation error $(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k})$.

2.2 Invariant Unscented Kalman filtering

This subsection is an extention of a previous research dealing with IUKF [11]. The motivation is that using the udapte equations of the IUKF algorithm we can specialize each step to make them more explicit in Section 4. If the dynamics of the observed system have invariance properties (symmetries) such as $f(\cdot)$ is G-invariant and $h(\cdot)$ is G-equivariant (see [2] for details), we cannot directly construct an estimator of the system state with analogous properties directly from the basic equations of the UKF algorithm. For convergence, it would be extremely desirable for any candidate estimator filter to satisfy the same invariance properties as the system itself, in the same spirit as the invariant observers of the IEKF algorithm. To achieve this, IUKF algorithm adapts the UKF algorithm so that it yields an invariant estimator. From the same principles and computation steps as the UKF algorithm, a *natural* reformulation of the equations aiming to adapt the method for estimation in an invariant setting can be obtained simply by redefining the error terms used of the standard algorithm. The linear state error $(\mathbf{x}_{k+1} - \hat{\mathbf{x}}_{k+1|k})$, the linear predicted output error $(\hat{\mathbf{y}}_{k+1|k} - \hat{\mathbf{y}}^{(i)}_{k+1|k})$ used in $\mathbf{P}^{\mathbf{y}}_{k+1|k}$ and $(\mathbf{y}_{k+1} - \hat{\mathbf{y}}_{k+1|k})$ conventionally used in Eq.(5) do not preserve any of the symmetries and invariance properties of the system. Instead, we consider in the IUKF algorithm the following invariant state error [1] and predicted output error on Lie group G such as $\forall \mathbf{g} \in G,\ \forall i \in [\![\, 0\,;2n\,]\!]$:

$$\begin{aligned} \boldsymbol{\eta}(\mathbf{x}_{k+1}, \hat{\mathbf{x}}_{k+1|k}) &= \mathbf{x}^{-1}_{k+1}\hat{\mathbf{x}}_{k+1|k} \\ \mathsf{E}(\hat{\mathbf{y}}_{k+1}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) &= \rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k}) \end{aligned} \tag{6}$$

Where $\mathbf{x}^{-1}_{k+1}$ is deduced from Cartan moving frame method and local transformation $\rho_{\mathbf{g}}$ is defined as for a dynamical system preserving symmetries [4]. An estimation $\hat{\mathbf{x}}_{k+1|k+1}$ of $\mathbf{x}_{k+1}$ is computed by the invariant Kalman filtering equations : $\hat{\mathbf{x}}_{k+1|k+1} = \hat{\mathbf{x}}_{k+1|k} + \mathbf{K}_{k+1}\mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}, \mathbf{y}_{k+1}, \omega(\hat{\mathbf{x}}_{k+1|k}))$, where $\omega(\hat{\mathbf{x}}_{k+1|k})$ is the standard basis of $\mathbb{R}^n$ formed by the invariant vector field $\mathcal{B}(\hat{\mathbf{x}}_{k+1|k}) = \{\omega_i(\hat{\mathbf{x}}_{k+1|k})\}_{i\in[\![\,1\,;n\,]\!]}$ (see [4] for more details). The unscented transfrom can be re-written in invariant form where the weighted sum of sigma point are written as equivalent invariant expressions.

Lemma 1. *(The invariant state error form of UT) : The unscented transform can be written with an invariant state error form as follow :*

$$\begin{aligned} \boldsymbol{\mathcal{X}}_{k+1|k} &= [\boldsymbol{\mathcal{X}}^{(0)}_{k+1|k}\ \ \boldsymbol{\mathcal{X}}^{(1)}_{k+1|k}\ \ \dots\ \ \boldsymbol{\mathcal{X}}^{(2n)}_{k+1|k}] = f(\boldsymbol{\mathcal{X}}_{k|k}, \mathbf{u}_k) \\ \hat{\mathbf{x}}_{k+1|k} &= \sum_{i=0}^{2n} W^{(i)}_{(m)}\, \boldsymbol{\mathcal{X}}^{(i)}_{k+1|k} \\ \mathbf{P}^{\mathbf{x}}_{k+1|k} &= \sum_{i=0}^{2n} W^{(i)}_{(c)} (\boldsymbol{\mathcal{X}}^{(i)^{-1}}_{k+1|k} \cdot \hat{\mathbf{x}}_{k+1|k})(\boldsymbol{\mathcal{X}}^{(i)^{-1}}_{k+1|k} \cdot \hat{\mathbf{x}}_{k+1|k})^T \end{aligned} \tag{7}$$

Lemma 2. *(The invariant output error form of UT) : The unscented transform can be written with an invariant output error form parametrized by the Lie group* $\mathbf{g}$ *as follow :*

$$\begin{aligned} \hat{\mathbf{Y}}_{k+1|k} &= [\hat{\mathbf{y}}^{(0)}_{k+1|k}\ \hat{\mathbf{y}}^{(1)}_{k+1|k}\ \dots\ \hat{\mathbf{y}}^{(2n)}_{k+1|k}] = h(\boldsymbol{\mathcal{X}}_{k+1|k}, \mathbf{u}_k) \\ \hat{\mathbf{y}}_{k+1|k} &= \sum_{i=0}^{2n} W^{(i)}_{(m)} \hat{\mathbf{y}}^{(i)}_{k+1|k} \\ \mathbf{P}^{\mathbf{y}}_{k+1|k} &= \sum_{i=0}^{2n} W^{(i)}_{(c)} E(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) E^T(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) \end{aligned} \tag{8}$$

Proof. See Appendix B.1 &B.2 □

[1] The group action coincides with left translations (resp. right translations), see [7] for details.

3 Problem setting

3.1 Considered Discret-Time model

We consider an Attitude and Heading Reference Systems (AHRS) in discrete-time [18] with step dt, characterized by its quaternion $\mathbf{q}_k$ with the quaternion product $*$. Eq (1) now becomes :

$$\begin{aligned} \mathbf{q}_{k+1} &= \mathbf{q}_k + 0.5.\mathbf{q}_k * (\boldsymbol{\omega}_{m_k} - \boldsymbol{\omega}_{b_k}).dt + \mathbf{w}_{\mathbf{q}_k} * \mathbf{q}_k \\ \boldsymbol{\omega}_{b_{k+1}} &= \boldsymbol{\omega}_{b_k} + \mathbf{q}_k{}^{-1} * dt.\mathbf{w}_{\mathbf{w_k}} * \mathbf{q}_k \\ a_{s_{k+1}} &= a_{s_k} + dt.\mathbf{w}_{a_k} \\ b_{s_{k+1}} &= b_{s_k} + dt.\mathbf{w}_{b_k} \end{aligned}$$

$$\begin{pmatrix} \mathbf{y}_{\mathbf{A}k} \\ \mathbf{y}_{\mathbf{B}k} \end{pmatrix} = \begin{pmatrix} a_{s_k}.\mathbf{q}_k^{-1} * \mathbf{A} * \mathbf{q}_k + \mathbf{v}_{\mathbf{A_k}} \\ b_{s_k}.\mathbf{q}_k^{-1} * \mathbf{B} * \mathbf{q}_k + \mathbf{v}_{\mathbf{B_k}} \end{pmatrix}, \tag{9}$$

Where $\mathbf{w}_{\mathbf{q_k}}, \mathbf{w}_{\mathbf{w_k}}, \mathbf{w}_{a_k}, \mathbf{w}_{b_k}$ (resp. $\mathbf{v}_{\mathbf{A_k}}, \mathbf{v}_{\mathbf{B_k}}$) refer to the process (resp. measurement) Gaussian white noise covariance matrices. The AHRS is endowed with 3 triaxial sensors: 3 magnetometers measure Earth's magnetic field, which is known constant and expressed in the body-fixed frame s.t. vector $\mathbf{y}_{\mathbf{B}_{k+1}} = \mathbf{q}^{-1}_{k+1} * \mathbf{B} * \mathbf{q}_{k+1}$ (where $\mathbf{B} = (B_x\ B_y\ B_z)^T$) can be considered as an output of the observation equations; 3 gyroscopes produce the measurements associated with the instantaneous angular velocities gathered in $\boldsymbol{\omega}_{m_k} \in \mathbb{R}^3$; and 3 accelerometers provide the measured output signals corresponding to the acceleration. Constant $\mathbf{A} = (0\ 0\ g)^T$ refers to the local Earth's gravity vector. Moreover, we add constant bias vector $\boldsymbol{\omega}_{b_k}$ on the angular velocities vector measurement $\boldsymbol{\omega}_{m_k}$ and constant scaling factor, denoted by a_{s_k} and b_{s_k}, which adjust and correct the predicted outputs $\mathbf{y}_{\mathbf{A}k}$ and $\mathbf{y}_{\mathbf{B}k}$. Note that the noise $\mathbf{w}_{\mathbf{w_k}}$ is defined as a Langevin noise i.e, this noise is said isotropic and enter into the system in an invariant way (see definition 1 in [15] for more details).

3.2 Invariance properties of the considered model

By taking advantage of the Galilean invariance properties of the problem, the model equations can be expressed equivalently in both aircraft coordinates and ground coordinates. Given Eq. (9) and the Lie group $G = \mathbb{H}_1 \times \mathbb{R}^5$ (where $\mathbb{H}_1$ is the Lie algebra of quaternions of norm one) acting on the entire state space, the dynamics of the system is indeed G-equivariant. We have :

Lemma 3. *Let G a Lie-group, $\forall \mathbf{g}_0 = (\mathbf{q}_0^T\ \boldsymbol{\omega}_0^T\ a_0\ b_0)^T \in G$, the following output transformation proves that system modeling is G-equivariant [3] : $\rho_{\mathbf{g}_0}(\mathbf{y}_{k+1}) = ((a_0.\mathbf{q}_0^{-1} * \mathbf{y}_{\mathbf{A}_{k+1}} * \mathbf{q}_0)^T\ (b_0.\mathbf{q}_0^{-1} * \mathbf{y}_{\mathbf{B}_{k+1}} * \mathbf{q}_0)^T)^T$.*

Moreover, the invariant state estimation error vector $\boldsymbol{\eta}(\mathbf{x}_{k+1}, \hat{\mathbf{x}}_{k+1|k})$, which is a transposition of the linear error to the multiplicative group may be defined by the following expression:

Lemma 4. *Consider* $(2n+1)$ *sigma points* $\boldsymbol{\mathcal{X}}$*, s.t.* $\boldsymbol{\mathcal{X}}^{(0)}_{k|k} = \hat{\mathbf{x}}_{k|k} = (\hat{\mathbf{q}}^T_{k|k}\ \hat{\boldsymbol{\omega}}^T_{b_{k|k}}\ \hat{a}_{s_{k|k}}\ \hat{b}_{s_{k|k}})$. *An invariant state estimation error* $\eta(\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}) = {\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}}^{-1} \cdot \hat{\mathbf{x}}_{k+1|k} / i \in [\![0\,; 2n]\!]$ *can be expressed by [3]*

$$\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \hat{\mathbf{x}}_{k+1|k}) = \begin{pmatrix} \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} * \hat{\mathbf{q}}_{k+1|k} \\ \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * (\boldsymbol{\omega}_{b,\boldsymbol{\mathcal{X}}^{(i)}} - \hat{\boldsymbol{\omega}}_{b,k+1|k}) * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \\ \hat{a}_{s_{k+1|k}} / a_{s,\boldsymbol{\mathcal{X}}^{(i)}} \\ \hat{b}_{s_{k+1|k}} / b_{s,\boldsymbol{\mathcal{X}}^{(i)}} \end{pmatrix} \tag{10}$$

Where:
$\boldsymbol{\mathcal{X}}^{(i)^{-1}} = (\mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}}\ \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \boldsymbol{\omega}_{b,\boldsymbol{\mathcal{X}}^{(i)}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}}\ (1/a_{s,\boldsymbol{\mathcal{X}}^{(i)}})\ (1/b_{s,\boldsymbol{\mathcal{X}}^{(i)}}))^T$.

The invariance properties of the IUKF applied on AHRS are closely intertwined with the invariant state estimation error. Along the line of the theorem 2 presented in [15], we consider the variable $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \hat{\mathbf{x}}_{k+1|k})$ as Markov processes, independent of the inputs $\boldsymbol{\omega}_{mk}$ [2]. The most important consequence of this property is that the invariant filter gain(s) calculation can be addressed *ad hoc* by choosing gain value(s) which will meet some predifined requirements in terms of : -convergence (guarantee and domain); - decoupling purposed (see [15] fot more details).

3.3 Toward an invariant unscented Kalman filter for AHRS

At this point, we should note that the algorithm presented in Section II uses a multiple parametrization of the transformation group obtained by successively defining the inverse of each sigma point as a parameter of the composite mapping $\phi_{\mathbf{g}} = (\mathbf{x}^{-1}_{k+1}, \rho_{\mathbf{g}})$. This is ultimately equivalent to defining a set of $(2n+1)$ n-dimensional moving frames in the state space, sending each sigma point to the identity element $\mathbf{e}$ via the local mapping $\mathbf{x}^{-1}_{k+1}$. The algorithm proposed here is generic in the sense that it does not assume any specific form for the equations of the observation model nor the relations which define the group transformation $\rho_{\mathbf{g}}$. Nevertheless, it can sometimes be useful to extend and specialize the computations in each of the steps listed above in order to make them more explicit for AHRS.

Problem 1. *The problem is to find a parametrization group* $\mathbf{g}$*, reducing the computation cost of the IUKF algorithm from Eq.(9) and the invariant predicted output error of unscented transform (See Lemma 2) such that :* $E(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) = \rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k})$.

We are now in a position to study the modification of the considered parametrization in the invariant predicted output error of the IUKF algorithm.

[2] When the consante biais vector $\boldsymbol{\omega}_{b_k}$ is correctly estimated that is the case in practice.

4 A parametric formulation study for AHRS

This section contains the main theoretical result of this article. This results, Theorem 2, consists of a formulation avoiding to find $4n^2+2n$, when the state dimension is n, invariant state errors between the sigma points. The main idea of the paper is to consider either $\mathbf{g_0} = {\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}}^{-1}$ or $\mathbf{g_0} = \hat{\mathbf{x}}^{-1}_{k+1|k}$. We futher make the following assumption :

Assumption 1. *Without loss of generality and emphasize the role of the parametrization, we assume evolution and observation noises in Eq.(9) equal to zero (i.e.,*$\mathbf{w}_k = \mathbf{v}_k = 0$*).*

4.1 The considered parametrization

4.1.1 Sigma-point as parametrization ($\mathbf{g_0} = {\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}}^{-1}$)

For AHRS, there is a straightforward way to express the $(2n+1)$ invariant output errors in terms of the constant vector $(\mathbf{A}^T\ \mathbf{B}^T)^T$, the transformation group $\rho_{\mathbf{g_0}}$, and a set of invariant state estimation errors satisfying the following relation for all $\forall i \in [\![0\,; 2n]\!]$:

Theorem 1. *Let* $(\mathbf{A}^T\ \mathbf{B}^T)^T$*, two constant vectors. For any sigma points* $\boldsymbol{\mathcal{X}}^{(i)}$*,* $\forall i \in [\![0\,; 2n]\!]$ *the predicted invariant output error* $E(\hat{\mathbf{y}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(i)}, \hat{\mathbf{y}}^{(i)}_{k+1|k})$*, denoted* $\hat{E}_{\boldsymbol{\mathcal{X}}}$ *for convenience can be expressed such as*

$$\hat{E}_{\boldsymbol{\mathcal{X}}} = \sum_{\substack{j=0 \\ j \neq i}}^{2n} W^{(j)}_{(m)} \left[\begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} - \rho_{\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \right]. \tag{11}$$

Proof. See Appendix C.1 □

Eq.(11) shows that the invariant prediction output errors can be written as a weighted sum of the (invariant) distances between the constant vector $(\mathbf{A}^T\ \mathbf{B}^T)^T$ and its image under the mapping $\rho_{\mathbf{g}}$ over the Lie group parametrized by the elements $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \boldsymbol{\mathcal{X}}^{(j)})$, where i ranges over $[\![0\,; 2n]\!]$ and $\forall j \in [\![0\,; 2n]\!]$. This expression requires to compute the values of the $(2n+1)$ invariant error vectors between the sigmas points $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \boldsymbol{\mathcal{X}}^{(j)})$. As a special case, whenever $i \in [\![0\,; 2n]\!]$, then $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \boldsymbol{\mathcal{X}}^{(i)}) = \vec{0}$.

Remark 1. *The algorithm therefore requires to find* $(2n+1) \times (2n+1) - (2n+1) = 4n^2 + 2n$ *terms* $\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)}, \boldsymbol{\mathcal{X}}^{(j)})$ *after these trivial cases are eliminated. In the simple case of an AHRS with* $n = 9$*, the IUKF approach therefore requires to compute* 342 *invariant error vectors between the sigma points at each iteration (i.e, prediction and correction step).*

For the AHRS, given a set of $(2n+1)$ sigma points, we implicitly need to compute a potentially large number of invariant estimation errors between the sigma points in order to compute the invariant output errors.

4.1.2 Predicted state as parametrization ($\mathbf{g_0} = \hat{\mathbf{x}}_{k+1|k}^{-1}$)

The parameter $\mathbf{g_0}$ of the group transformation $\rho_{\mathbf{g_0}}$can also be chosen to be constant and equal to $\hat{\mathbf{x}}_{k+1|k}$.The invariant output predicted errors can be expressed as :

Theorem 2. *Solution to problem : Let $(\mathbf{A}^T\ \mathbf{B}^T)^T$, two constant vectors. For any predicted state $\hat{\mathbf{x}}_{k+1|k}, \forall i \in [\![0\,;2n]\!]$ the predicted invariant output error $E(\hat{\mathbf{y}}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}, \hat{\mathbf{y}}^{(i)}_{k+1|k})$, denoted $\hat{E}_{\hat{\mathbf{x}}}$ for convenience, can be expressed such as*

$$\hat{E}_{\hat{\mathbf{x}}} = \rho_{\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(i)},\hat{\mathbf{x}}_{k+1|k})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)}\rho_{\boldsymbol{\eta}(\boldsymbol{\mathcal{X}}^{(j)},\hat{\mathbf{x}}_{k+1|k})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}. \tag{12}$$

Proof. See Appendix C.2 □

The significance of this result is that each elementary error term takes an invariant of the estimation problem as an argument, namely the constant vector $(\mathbf{A}^T\ \mathbf{B}^T)^T$, and that the parametrization of the Lie group ranges over the index j of the weighted sum, depending on the sigma point considered in each elementary calculation.

Remark 2. *Unlike the earlier case, we only need to know $2n = 18$ invariant state estimation errors between the predicted state and each sigma point in order to determine the errors in the predicted outputs.*

4.2 Invariant unscented Kalman filter equations for AHRS

In terms of the unscented transform $\mathrm{UT}(\cdot)$ the *invariant unscented Kalman filter for AHRS* prediction and update steps can be written by using Lemma 4. and Theorem 2. as

- *Prediction:* Compute the predicted state mean $\hat{\mathbf{x}}_{k+1|k}$ and the predicted covariance $\mathbf{P}^{\mathbf{x}}_{k+1|k}$ as
$$\begin{aligned}[\hat{\mathbf{x}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k}] &= \mathrm{UT}(f, \hat{\mathbf{x}}_{k|k}, \mathbf{P}^{\mathbf{x}}_{k|k}, \mathbf{x}_k^{-1}\hat{\mathbf{x}}_{k|k})\\ \mathbf{P}^{\mathbf{x}}_{k+1|k} &= \tilde{\mathbf{P}}^{\mathbf{x}}_{k+1|k} + \mathbf{W}_k\end{aligned} \tag{13}$$
- *Update:* Compute the predicted mean $\hat{\mathbf{y}}_{k+1|k}$ and covariance of the measurement $\mathbf{P}^{\mathbf{y}}_{k+1|k}$, and the cross-corvariance of the state and measurement $\mathbf{P}^{\mathbf{xy}}_{k+1|k}$:
$$\begin{aligned}[\hat{\mathbf{y}}_{k+1|k}, \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k}] &= \mathrm{UT}(h, \hat{\mathbf{x}}_{k+1|k}, \mathbf{P}^{\mathbf{x}}_{k+1|k}, \hat{E}_{\hat{\mathbf{x}}})\\ \mathbf{P}^{\mathbf{y}}_{k+1|k} &= \tilde{\mathbf{P}}^{\mathbf{y}}_{k+1|k} + \mathbf{V}_k\\ \mathbf{P}^{\mathbf{xy}}_{k+1|k} &\propto (\mathbf{x}_{k+1}^{-1}\hat{\mathbf{x}}_{k+1|k}, \hat{E}_{\hat{\mathbf{x}}})\end{aligned} \tag{14}$$

An estimation $\hat{\mathbf{x}}_{k+1|k+1}$ of $\mathbf{x}_{k+1}$ is then computed by the Kalman filtering equations using Eq.(12) :

$$\hat{\mathbf{x}}_{k+1|k+1} = \sum_{j=0}^{2n} W^{(j)}_{(m)}\Bigg[\boldsymbol{\mathcal{X}}^{(j)}_{k+1|k}\cdots + \sum_{i=1}^{n}\mathbf{K}^{(i)}_{k+1}\left(\rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\mathbf{y}_{k+1}) - \rho_{\boldsymbol{\eta}(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})}\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}\right)\omega_i(\hat{\mathbf{x}}_{k+1|k})\Bigg] \tag{15}$$

Proof. See Appendix C.3 □

Fig. 1. Pictorial overview of SPQR concept.

5 Simulation results

The motivation for this simulation is led by the development of Small Payload Quick-Return (SPQR) study intended to routinely deliver small payloads from International Space Station (ISS) on-demand [3]. The SPQR concept, originating from NASA Ames Research Center at Moffett Field, CA, relies on a three-stage method of returning payloads, after being stored until needed and then loaded while on-board the ISS (cf. Figure 1): a) Deorbit, by means of a passive deployable drag system; b) Atmospheric reentry, via the deployment of a passively self-stabilizing reentry body; c) Terminal descent of the temperature-controlled paylaod canister beneath an autonomous guided parafoil. To mature this final phase of the SPQR concept, an autonomous parafoil system which satisfies the demands of landing precision requirements must be developed by using a small AHRS payload. We illustrate the behavior of the UKF and the IUKF dedicated to AHRS with both parametrizations ($\mathrm{IUKF}_{\mathbf{x}}$,$\mathrm{IUKF}_{\boldsymbol{\mathcal{X}}}$) by simulations and compare this one with experiment data. A set of results for the AHRS estimation problem generated from simulated noisy data are then presented to demonstrate the well-foundedness of the IUKF algorithm using state as parametrization ($\mathrm{IUKF}_{\mathbf{x}}$), as well as its potential benefits in both theoretical and SPQR contexts.

5.1 Simulation Setting

The reference input data used for our evaluation of an IUKF-type approach were generated by dynamic model simulations describing the free fall of a parafoil. These simulated data provide a straightforward way to validate the methodological principles presented in this article, configure the parameters of each method, and establish a few preliminary conclusions regarding the analysis on the computational effort. We considered the data both with and without added noise. The reference simulation that we used to validate our algorithms had a duration of slightly over 100 seconds. The simulated parachute system exhibits relatively strong dynamics, as can be seen in the figure 2. The roll, heading, and pitch angles vary by up to several dozen degrees. The UAV also experienced significant variations in the velocity, partially invalidating one of the hypotheses of the model in Eq.(9), namely the assumption that the linear acceleration is negligible i.e, $\dot{V} = 0$ (See Figure 2). It would therefore be interesting to investigate the effects of the error introduced into the estimation process by assuming that $\dot{V} = 0$.

[3] https://www.nasa.gov/mission_pages/station/research/experiments/2543.html

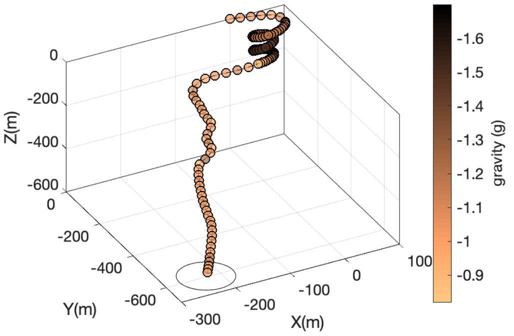


Fig. 2. 3D trajectory of the parafoil in terminal descent starting from the position (0,0,0). The simulated parachute system exhibits relatively strong dynamics. The linear acceleration $\dot{V}$ is non-zero from (0,0,0) to (0,-200,-200).

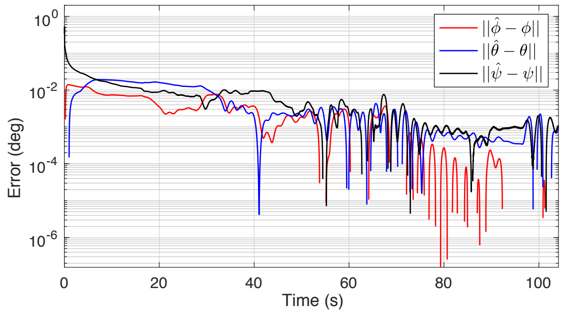
Analysis of the simulated data shows that $\dot{V}$ was non-zero throughout the period $t \in [5\,;40]$. The results and estimates obtained below by applying both IUKF algorithms to the data are compared against results from the standard UKF algorithm in each case.

5.2 Noise-Free Simulations

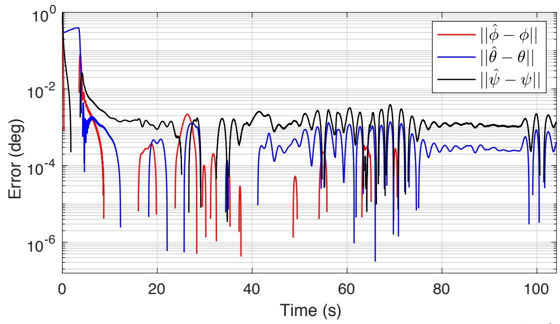
To emphase the effect of the parametrization on algorithms performance, we first assume that the sensors are perfect (see Assumption 1), i.e., without noise. Figure 3 shows the estimated attitude errors computed by the UKF, IUKF$_{\boldsymbol{\mathcal{X}}}$ and IUKF$_{\mathbf{x}}$ algorithms. Each estimate is compared against the pseudo-measurements reconstructed from the components of the reference quaternion state vector. The estimated angles match the reference values almost perfectly. The attitude of the parafoil is correctly reconstruct with respect to all three axes. Note that the error introduced into the initial state of the simulation was corrected very rapidly, after only a few computation steps (characteristic time $<$ 0.5 sec.). However, the estimation errors (with a log scale along the vertical axis) show that the IUKF estimator converges more closely and quickly to the true values of the flight parameters. The IUKF$_{\mathbf{x}}$ achieves smaller estimation errors than the UKF and IUKF$_{\boldsymbol{\mathcal{X}}}$. Additionally, the comparison of these error plots suggests that the residuals of the state estimate constructed by IUKF$_{\mathbf{x}}$ appear to be more stable over time; a slight albeit slow decrease of these residuals may be observed in the results generated by the UKF algorithm due to $\dot{V} \neq 0$ throughout the period $t \in [5\,;40]$. The estimates of the new IUKF parametrization therefore outperforms the standard variant of the UKF algorithm.

5.3 Measurement Noise

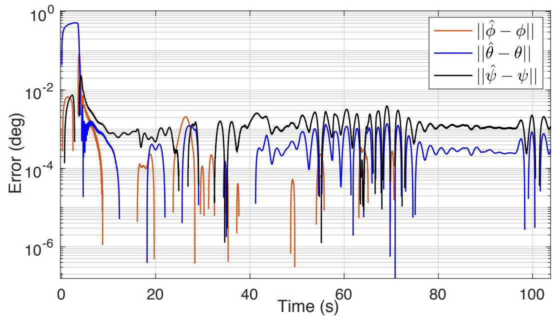
We now study the impact of the measurement noise on the algorithms performance. Here, a series of additive colored noise terms were incorporated into the reference simulation as perturbations of the measurements $\boldsymbol{\omega}_m$, y_{Am}, and y_{Bm}. The experimental data are sampled with a frequency equal to 50Hz which characterized the inertial measurement unit and the magnetometers. The measurements are corrupted by gaussian white noises whose standard deviations are set to :



(a) UKF : estimated errors of the attitude anlges (ϕ, θ, ψ).



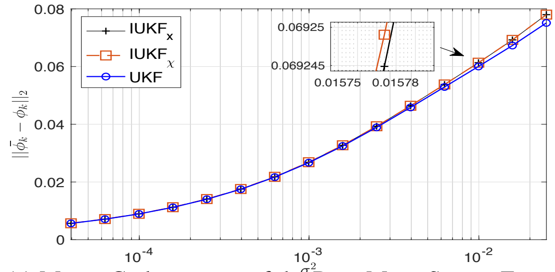
(b) IUKF$_{\boldsymbol{\mathcal{X}}}$: estimated errors of the attitude anlges (ϕ, θ, ψ).



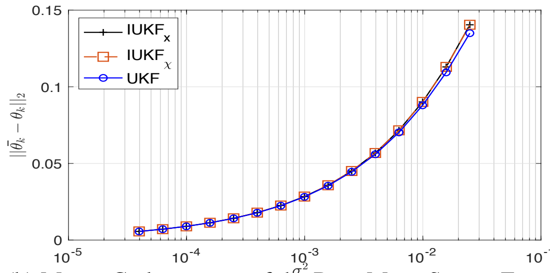
(c) IUKF$_{\mathbf{x}}$: estimated errors of the attitude anlges (ϕ, θ, ψ).

Fig. 3. Attitude estimation errors with wrong initial angles and $\dot{V} \neq 0$ throughout the period $t \in [5\,;40]$. The plots show that the IUKF$_{\mathbf{x}}$ estimator converge more closely and quikly to the true values of the flight parameters.

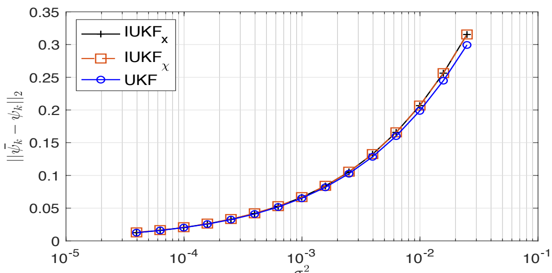
$\sigma_{\text{gyro}} = 0.2\ {}^\circ/s$, $\sigma_{\text{accelero}} = 0.2$ g and $\sigma_{\text{magneto}} = 500$ nT. In order to validate our filters, we have also introduced a biases vector on $\boldsymbol{\omega}_m$ s.t. $\boldsymbol{\omega}_b = [0.1 \text{ rad/s } 0.05 \text{ rad/s } 0.02 \text{ rad/s}]^T$. The two positive scalar factor values are set to $a_s = 1.2$ and $b_s = 0.9$ respectively. Initially, we set for the filters with incorrect angles $\bar{\phi} \sim \mathcal{N}\Big(0, (\pi/4)^2\Big), \bar{\theta} \sim \mathcal{N}\Big(0, (\pi/4)^2\Big), \bar{\psi} \sim \mathcal{N}\Big(0, (\pi/2)^2\Big)$. We then run 500 Monte-Carlo simulations for different levels of measurement noise $\sigma^2_{\text{accelero}} = [4.10^{-4}, 3.10^{-2}]$ and compare the (average) Root Mean Square Errors w.r.t the reference values over the whole trajectory. Figures 4 compare the results produced by the IUKF$_{\mathbf{x}}$,IUKF$_{\boldsymbol{\mathcal{X}}}$ and UKF algorithms. For this trajectory, we see that IUKF$_{\mathbf{x}}$ is sightly better than IUKF$_{\boldsymbol{\mathcal{X}}}$. The UKF is sightly better than IUKF when noise is moderate.



(a) Monte-Carlo average of the Root Mean Square Error on $(\phi_k)_{1\leqslant k\leqslant N}$ over the whole trajectory, as a function of the noise measurement variance σ^2.



(b) Monte-Carlo average of the Root Mean Square Error on $(\theta_k)_{1\leqslant k\leqslant N}$ over the whole trajectory, as a function of the noise measurement variance σ^2.



(c) Monte-Carlo average of the Root Mean Square Error on $(\psi_k)_{1\leqslant k\leqslant N}$ over the whole trajectory, as a function of the noise measurement variance σ^2.

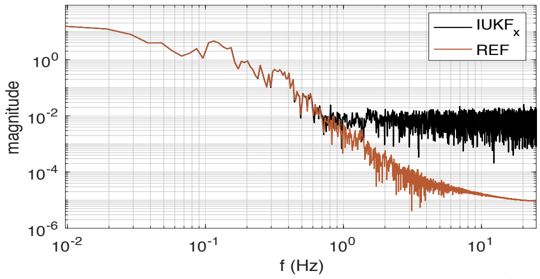
Fig. 4. Noisy case - comparison of the average Root Mean Square Error of IUKF$_{\mathbf{x}}$,IUKF$_{\boldsymbol{\chi}}$ and UKF. The IUKF$_{\mathbf{x}}$ is sightly better than IUKF$_{\boldsymbol{\chi}}$ algorithm but the UKF is sightly better than IUKF when noise is moderate.

5.4 *Study in the frequency domain*

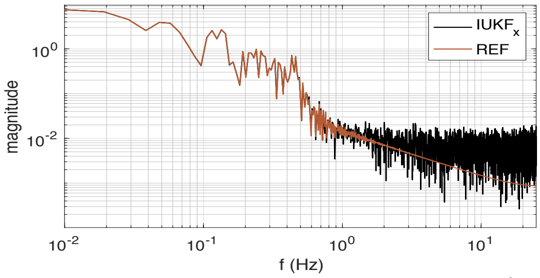
In order to analyse the bandwith performance of the IUKF$_{\mathbf{x}}$ algorithm, we performed a frequency analysis based on FFT of the estimated signal. The frequency analysis shows that the reference frequencies are tracked over a broad spectrum of frequencies, i.e. up to 0.7 Hz equivalent to 252 $^\circ/s$ for pitch and roll which is less than the maximun angular velocity used in our application.

5.5 *Initial analysis on the computational effort*

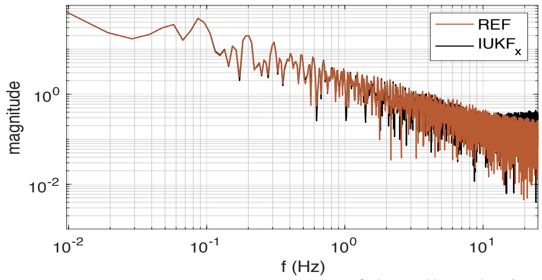
In case of a real time embedded application, another interesting filter characteristic is the computational effort. In section 4, we claimed the reduction of the computational burden of the IUKF parametrize by the state $\mathbf{x}_k$. A first computational analysis consist in computing the average time calculation using 500 Monte-Carlo simulation using an Intel Core I7 2.Ghz. We thus see in the table below that the computation time of the IUKF is slightly higher ($\sim$19.5%) than UKF



(a) Frequency response of the roll angle ϕ.



(b) Frequency response of the roll angle θ.



(c) Frequency response of the roll angle ψ.

Fig. 5. Study of attitude estimation of the IUKF parametrized by the state in the frequency domain.

algorithm due to additional operation on invariant state error. But results clearly reveal that the IUKF parametrized by the state is more computation time efficient ($\sim$5.1%) than IUKF parametrized by sigma-point. In future work, the memory allocation, which is a relevant parameter for embedded system, will be studied for each algorithms.

Filter	Computation time (s)	RMSE (s)
① UKF	0.56 ($\Delta_{①\to③}\sim +19.5\%$)	1e-4
② IUKF$_{\boldsymbol{\chi}}$	0.7345	1.6e-4
③ IUKF$_{\mathbf{x}}$	0.6988($\Delta_{②\to③}\sim -5.1\%$)	1.6e-4

6 Conclusion and perspectives

The presented algorithm is combining efficiently the invariant observers and the non-linear unscented filtering theories. The advantage is that is does not require to linearize the problem while taking into account dynamical symmetries. In the case of the attitude estimation problem, it has been demonstrated that a specific formulation can reduce the computation cost without compromising the stability and precision of the filter. Future work will include performance test on embedded micro-controllers for real-world applications.

A Unscented transfrom

The *unscented transform* [17] can be used for forming a Gaussian approximation to the joint distribution of random variable $\mathbf{x}_{k|k}$ and $\mathbf{y}_{k|k}$, when the random variable $\mathbf{y}_{k|k}$ is obtained by a non-linear transformation of the Gaussian random variable $\mathbf{x}_{k|k}$ as follow :

$$\begin{cases} \mathbf{x}_{k|k} \sim N(\hat{\mathbf{x}}_{k|k}, \mathbf{P}_{k|k}) \\ \mathbf{y}_{k|k} = \gamma(\mathbf{x}_{k|k}, k) \end{cases} \tag{A.1}$$

The aims of the basic UT is to form a fixed number of deterministically chosen sigma-points $\boldsymbol{\mathcal{X}}_{k|k}$, which capture the "true" mean $\hat{\mathbf{x}}_{k|k}$ and covariance $\mathbf{P}_{k|k}$ of the original distribution $\mathbf{x}_{k|k}$. This set of points must represent accurately the first and second order moments. These sigma-points are then propagated through the nonlinear functions Eq.(A.1) providing a cloud of evolving points. The mean $\hat{\mathbf{x}}_{k|k+1}$ and estimated covariance matrix $\mathbf{P}_{k|k+1}$ of the transformed points are then computed based on their statistics. The unscented transform can be used for forming

$$\begin{pmatrix} \mathbf{x}_{k|k} \\ \mathbf{y}_{k|k} \end{pmatrix} \sim N\left(\begin{pmatrix} \hat{\mathbf{x}}_{k|k} \\ \hat{\mathbf{y}}_{k+1|k} \end{pmatrix}, \begin{pmatrix} \mathbf{P}^{\mathbf{x}}_{k|k} & \mathbf{P}^{\mathbf{xy}}_{k|k} \\ \mathbf{P}^{\mathbf{yx}}_{k|k} & \mathbf{P}^{\mathbf{y}}_{k|k} \end{pmatrix} \right) \tag{A.2}$$

to the joint probability density of $\mathbf{x}_{k|k} \in \mathbb{R}^n$ and $\mathbf{y}_{k|k} \in \mathbb{R}^m$. The unscented transform is the following :

(1) Form the set of $2n+1$ sigma points from the coloms of the $n \times n$ matrix $(\sqrt{n+\lambda})P_{k|k}$ as follows:

$$\begin{cases} \boldsymbol{\mathcal{X}}^{(0)}_{k|k} = \mathbf{x}_{k|k} \\ \boldsymbol{\mathcal{X}}^{(i)}_{k|k} = \mathbf{x}_{k|k} + \left[\sqrt{(n+\lambda)\mathbf{P}^{\mathbf{x}}_{k|k}}\right], i = 1, \cdots, n \\ \boldsymbol{\mathcal{X}}^{(i)}_{k|k} = \mathbf{x}_{k|k} - \left[\sqrt{(n+\lambda)\mathbf{P}^{\mathbf{x}}_{k|k}}\right], i = n+1, \cdots, 2n \end{cases} \tag{A.3}$$

and compute the associated weights $W_m^{(0)}$,$W_c^{(0)}$,$W_m^{(i)}$,$W_c^{(i)}$ [4].

(2) Transform each of sigma as

$$\mathbf{y}^{(i)}_{k+1|k} = \gamma(\boldsymbol{\mathcal{X}}^{(i)}_{k|k}), i = 0, \cdots, 2n. \tag{A.4}$$

(3) Mean and covariance estimates for $\mathbf{y}_{k+1|k}$ can be computed as

$$\hat{\mathbf{y}}_{k+1|k} \approx \sum_{i=0}^{2n} W_m^{(i)} \hat{\mathbf{y}}^{(i)}_{k+1|k} \tag{A.5}$$

$$\mathbf{P}^{\mathbf{y}}_{k+1|k} \approx \sum_{i=0}^{2n} W_c^{(i)} (\hat{\mathbf{y}}^{(i)}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k})(\hat{\mathbf{y}}^{(i)}_{k+1|k} - \hat{\mathbf{y}}_{k+1|k})^T \tag{A.6}$$

[4] $W_m^{(0)} = \lambda/(n+\lambda), W_c^{(0)} = \lambda/(n+\lambda) + (1-\alpha^2+\beta), W_m^{(i)} = 1/\{2(n+\lambda)\}, i = 1, \cdots, 2n, W_c^{(i)} = 1/\{2(n+\lambda)\}, i = 1, \cdots, 2n$. The parameter λ is a scaling parameter defined as $\lambda = \alpha^2(n+\kappa) - n$. The positive constants α, β and κ are used as parameters of the method. We set uncented transform parameters to $\kappa = 0$ and $\beta = 2$. α keeps a free-parameter chosen by the practitioner, which must be small ($\alpha = 10^{-3}$ in our applications).

B Derivations

B.1 Derivation of the invariant state error form of UT

Let's consider a group action, full-rank and transitive (i.e. $\dim(G) = \dim(\mathcal{X}) = n$). G can be identified with the state space $\mathcal{X} = \mathbb{R}^n$ in such a way that the local transformation on the state $\varphi_{\mathbf{g}}$ is viewed as the left or right-multiplication mapping $\varphi_{\mathbf{g}}(\mathbf{x}) = \mathbf{g} \cdot \mathbf{x}$. Solving the normalization equations to obtain $\varphi_{\mathbf{g}}(\mathbf{x}) = \mathbf{g} \cdot \mathbf{x} = e$ from Cartan moving frame method, where e is the identity element of the group G, gives us the moving frame $\gamma(\mathbf{x}) = \mathbf{x}^{-1}$ as a solution [**?**]. If we define the invariant state error on Lie group G such as

$$\boldsymbol{\eta}(\mathbf{x}_{k+1}, \hat{\mathbf{x}}_{k+1|k}) = \mathbf{x}^{-1}_{k+1} . \hat{\mathbf{x}}_{k+1|k} \tag{B.1}$$

then the unscented transform in Eq.(3) can be written in form of the last equation in Eq.(7).

$$\boldsymbol{\mathcal{X}}_{k+1|k} = [\boldsymbol{\mathcal{X}}^{(0)}_{k+1|k} \ \ \boldsymbol{\mathcal{X}}^{(1)}_{k+1|k} \ \ \ldots \ \ \boldsymbol{\mathcal{X}}^{(2n)}_{k+1|k}] = f(\boldsymbol{\mathcal{X}}_{k|k}, \mathbf{u}_k)$$

$$\hat{\mathbf{x}}_{k+1|k} \approx \sum_{i=0}^{2n} W^{(i)}_{(m)} \boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}$$

$$\begin{aligned} \mathbf{P}^{\mathbf{x}}_{k+1|k} &\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} (\varphi_{\boldsymbol{\mathcal{X}}^{(i)-1}_{k+1|k}}(\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}) - \varphi_{\boldsymbol{\mathcal{X}}^{(i)-1}_{k+1|k}}(\hat{\mathbf{x}}_{k+1|k})) \\ &\quad \times (\varphi_{\boldsymbol{\mathcal{X}}^{(i)-1}_{k+1|k}}(\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}) - \varphi_{\boldsymbol{\mathcal{X}}^{(i)-1}_{k+1|k}}(\hat{\mathbf{x}}_{k+1|k}))^T \\ &\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} \eta(\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}) \eta^T(\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}, \hat{\mathbf{x}}_{k+1|k}) \\ &\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} (\boldsymbol{\mathcal{X}}^{(i)^{-1}}_{k+1|k} \cdot \hat{\mathbf{x}}_{k+1|k})(\boldsymbol{\mathcal{X}}^{(i)^{-1}}_{k+1|k} \cdot \hat{\mathbf{x}}_{k+1|k})^T \end{aligned}$$

B.2 Derivation of the invariant output error form of UT

If we define the invariant output error using the Lie group $\forall \mathbf{g} \in G$ such as

$$\mathsf{E}(\hat{\mathbf{y}}_{k+1}, \mathbf{g}, \hat{\mathbf{y}}^{(i)}_{k+1|k}) = \rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k}) \tag{B.2}$$

then the unscented transform in Eq.(4) can be written in form of the last equation in Eq.(8).

$$\hat{\mathbf{Y}}_{k+1|k} = [\hat{\mathbf{y}}^{(0)}_{k+1|k} \ \hat{\mathbf{y}}^{(1)}_{k+1|k} \ \ldots \ \hat{\mathbf{y}}^{(2n)}_{k+1|k}] = h(\boldsymbol{\mathcal{X}}_{k+1|k}, \mathbf{u}_k)$$

$$\hat{\mathbf{y}}_{k+1|k} \approx \sum_{i=0}^{2n} W^{(i)}_{(m)} \hat{\mathbf{y}}^{(i)}_{k+1|k}$$

$$\begin{aligned} \mathbf{P}^{\mathbf{y}}_{k+1|k} &\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} (\rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k})) \\ &\quad \times (\rho_{\mathbf{g}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\mathbf{g}}(\hat{\mathbf{y}}^{(i)}_{k+1|k}))^T \\ &\approx \sum_{i=0}^{2n} W^{(i)}_{(c)} \mathsf{E}(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \mathbf{y}^{(i)}_{k+1|k}) \\ &\quad \times \mathsf{E}^T(\hat{\mathbf{y}}_{k+1|k}, \mathbf{g}, \mathbf{y}^{(i)}_{k+1|k}) \end{aligned}$$

which leads to last equation in Lemma 1 and Lemma 2.

C Proofs of the results of section V

C.1 Proof of Theorem 1

Let $\mathbf{g_0} = {\boldsymbol{\mathcal{X}}^{(i)}_{k+1|k}}^{-1}$, we have :

$$
\begin{aligned}
\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}} &= \rho_{\boldsymbol{\mathcal{X}}^{(i)-1}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\boldsymbol{\mathcal{X}}^{(i)-1}}(\hat{\mathbf{y}}^{(i)}_{k+1|k}) \\
&= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \begin{pmatrix} a^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \hat{\mathbf{y}}_{\mathbf{A}_{k+1|k}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \\ b^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \hat{\mathbf{y}}_{\mathbf{B}_{k+1|k}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \end{pmatrix} \\
&= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \cdots \\
&\quad - \begin{pmatrix} a^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \Big(\sum_{j=0}^{2n} W^{(j)}_{(m)} h_{\mathbf{A}}(\boldsymbol{\mathcal{X}}^{(j)}) \Big) * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \\ b^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \Big(\sum_{j=0}^{2n} W^{(j)}_{(m)} h_{\mathbf{B}}(\boldsymbol{\mathcal{X}}^{(j)}) \Big) * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \end{pmatrix}
\end{aligned}
$$

Where $h_{\mathbf{A}}(\cdot)$ and $h_{\mathbf{B}}(\cdot)$ denotes the restriction of the observation model to the outputs associated with the acceleration and the magnetic field respectively.

$$
\begin{aligned}
\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}} &= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)} \cdots \\
&\times \begin{pmatrix} a^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \big(a_{s,\boldsymbol{\mathcal{X}}^{(j)}} \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{A} * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} \big) * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \\ b^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \big(b_{s,\boldsymbol{\mathcal{X}}^{(j)}} \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{B} * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} \big) * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} \end{pmatrix} \\
&= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)} \cdots && \text{(C.1)} \\
&\times \begin{pmatrix} a^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} a_{s,\boldsymbol{\mathcal{X}}^{(j)}} \cdot (\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}})^{-1} * \dots \\ b^{-1}_{s,\boldsymbol{\mathcal{X}}^{(i)}} b_{s,\boldsymbol{\mathcal{X}}^{(j)}} \cdot (\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}})^{-1} * \dots \end{pmatrix} && \text{(C.2)} \\
&\qquad \begin{matrix} \mathbf{A} * (\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}}) \\ \mathbf{B} * (\mathbf{q}_{\boldsymbol{\mathcal{X}}^{(j)}} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}}) \end{matrix} \Bigg) && \text{(C.3)} \\
&= \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\eta(\boldsymbol{\mathcal{X}}^{(i)},\boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}
\end{aligned}
$$

We have

$$\sum_{j=0}^{2n} W^{(j)}_{(m)} = 1.$$

Hence

$$\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}} = \sum_{j=0}^{2n} W^{(j)}_{(m)} \left[\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \rho_{\eta(\boldsymbol{\mathcal{X}}^{(i)},\boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \right] \tag{C.4}$$

Moreover

$$\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \rho_{\eta(\boldsymbol{\mathcal{X}}^{(i)},\boldsymbol{\mathcal{X}}^{(i)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} = \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \rho_{\vec{0}} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} = \vec{0}$$

Equation (C.4) can be expressed in the case where $j \neq i$, which concludes the proof with $\forall i \in [\![\, 0\,; 2n \,]\!]$:

$$\hat{\mathsf{E}}_{\boldsymbol{\mathcal{X}}} = \sum_{\substack{j\,=\,0 \\ j\,\neq\, i}}^{2n} W^{(j)}_{(m)} \left[\begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \rho_{\eta(\boldsymbol{\mathcal{X}}^{(i)},\boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \right].$$

C.2 Proof of Theorem 2

Let $\mathbf{g_0} = \hat{\mathbf{x}}^{-1}_{k+1|k}$. We have

$$
\begin{aligned}
\hat{\mathsf{E}}_{\hat{\mathbf{x}}} &= \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\hat{\mathbf{y}}_{k+1|k}) - \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\hat{\mathbf{y}}^{(i)}_{k+1|k}) \\
&= \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(h_{\mathbf{A},\mathbf{B}}(\boldsymbol{\mathcal{X}}^{(i)})) - \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(\hat{\mathbf{y}}_{k+1|k}) \\
&= \rho_{\hat{\mathbf{x}}^{-1}_{k+1|k}}(h_{\mathbf{A},\mathbf{B}}(\boldsymbol{\mathcal{X}}^{(i)})) - \cdots && \text{(C.5)} \\
&\qquad \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} \\
&= h_{\mathbf{A},\mathbf{B}}(\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(i)})) - \cdots && \text{(C.6)} \\
&\qquad \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}
\end{aligned}
$$

With $h_{\mathbf{A},\mathbf{B}}(\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(i)}))$, we have :

$$\underbrace{\begin{pmatrix} a_{s,\boldsymbol{\mathcal{X}}^{(i)}} \cdot \hat{a}^{-1}_{s,k+1|k} \hat{\mathbf{q}}_{k+1|k} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} * \mathbf{A} * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \hat{\mathbf{q}}^{-1}_{k+1|k} \\ b_{s,\boldsymbol{\mathcal{X}}^{(i)}} \hat{b}^{-1}_{s,k+1|k} \cdot \hat{\mathbf{q}}_{k+1|k} * \mathbf{q}^{-1}_{\boldsymbol{\mathcal{X}}^{(i)}} * \mathbf{B} * \mathbf{q}_{\boldsymbol{\mathcal{X}}^{(i)}} * \hat{\mathbf{q}}^{-1}_{k+1|k} \end{pmatrix}.}_{= \qquad \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(i)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}} \tag{C.7}$$

The claimed predicted invariant output error can be expressed as a weighted sum of invariant output errors such as:

$$\hat{\mathsf{E}}_{\hat{\mathbf{x}}} = \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(i)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix} - \sum_{j=0}^{2n} W^{(j)}_{(m)} \rho_{\eta(\hat{\mathbf{x}}_{k+1|k},\boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix}\mathbf{A}\\ \mathbf{B}\end{pmatrix}.$$

C.3 Derivation of the IUKF$_{\mathbf{x}}$ for the AHRS

As shown in Section V, predicted state parametrization $\hat{\mathsf{E}}_{\mathbf{x}}$ avoids the need to find $4n^2 + 2n$ invariant state errors. The invariant unscented Kalman filter equations of the AHRS problem can be derived as follows.

$$
\begin{aligned}
\hat{\mathbf{x}}_{k+1|k+1} &= \hat{\mathbf{x}}_{k+1|k} + \sum_{i=1}^{n} \mathbf{K}_{k+1}^{(i)} \cdots \\
&\times \mathsf{E}(\mathbf{y}_{k+1}, \hat{\mathbf{x}}_{k+1|k}, \hat{\mathbf{y}}_{k+1|k}) \cdot w_i(\hat{\mathbf{x}}_{k+1|k}) \\
&= \hat{\mathbf{x}}_{k+1|k} + \sum_{i=1}^{n} \mathbf{K}_{k+1}^{(i)} \big(\rho_{\hat{\mathbf{x}}_{k+1|k}^{-1}}(\mathbf{y}_{k+1}) \cdots \\
&\quad -\rho_{\hat{\mathbf{x}}_{k+1|k}^{-1}}(\hat{\mathbf{y}}_{k+1|k})\big) \cdot w_i(\hat{\mathbf{x}}_{k+1|k}) \\
&= \hat{\mathbf{x}}_{k+1|k} + \sum_{i=1}^{n} \mathbf{K}_{k+1}^{(i)} \Bigg(\rho_{\hat{\mathbf{x}}_{k+1|k}^{-1}}(\mathbf{y}_{k+1}) \cdots \\
&\quad - \sum_{j=0}^{2n} W_{(m)}^{(j)} \rho_{\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \Bigg) w_i(\hat{\mathbf{x}}_{k+1|k}) \\
&= \hat{\mathbf{x}}_{k+1|k} + \sum_{i=1}^{n} \mathbf{K}_{k+1}^{(i)} \sum_{j=0}^{2n} W_{(m)}^{(j)} \Bigg(\rho_{\hat{\mathbf{x}}_{k+1|k}^{-1}}(\mathbf{y}_{k+1}) \\
&\qquad -\rho_{\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \Bigg) w_i(\hat{\mathbf{x}}_{k+1|k})
\end{aligned}
$$

At each time step, the estimated state is therefore computed in the form of a correction of the prediction derived from the left or right-invariant dynamics of the system expressed as a weighted sum of invariant innovation terms over the Lie group parametrized directly or indirectly by $\hat{\mathbf{x}}_{k+1|k+1}$.

$$
\begin{aligned}
&= \hat{\mathbf{x}}_{k+1|k} + \sum_{j=0}^{2n} W_{(m)}^{(j)} \sum_{i=1}^{n} \mathbf{K}_{k+1}^{(i)} \Bigg(\rho_{\hat{\mathbf{x}}_{k+1|k}^{-1}}(\mathbf{y}_{k+1}) \cdots \\
&\quad -\rho_{\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \Bigg) \cdot w_i(\hat{\mathbf{x}}_{k+1|k}) \\
&= \hat{\mathbf{x}}_{k+1|k} + \sum_{j=0}^{2n} W_{(m)}^{(j)} \Bigg[\boldsymbol{\mathcal{X}}_{k+1|k}^{(j)} + \sum_{i=1}^{n} \mathbf{K}_{k+1}^{(i)} \Bigg(\rho_{\hat{\mathbf{x}}_{k+1|k}^{-1}}(\mathbf{y}_{k+1}) \\
&\qquad -\rho_{\eta(\hat{\mathbf{x}}_{k+1|k}, \boldsymbol{\mathcal{X}}^{(j)})} \begin{pmatrix} \mathbf{A} \\ \mathbf{B} \end{pmatrix} \Bigg) \cdot w_i(\hat{\mathbf{x}}_{k+1|k}) \Bigg]
\end{aligned}
$$

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