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► **To cite this version:**

| Nicolas Couellan. Probabilistic Robustness Estimates for Deep Neural Networks. 2020. hal-02572277

**HAL Id: hal-02572277**

**<https://hal-enac.archives-ouvertes.fr/hal-02572277>**

Preprint submitted on 13 May 2020

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# Probabilistic Robustness Estimates for Deep Neural Networks

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May 13, 2020

## Abstract

Robustness of deep neural networks is a critical issue in practical applications. In the case of deep dense neural networks, under random noise attacks, we propose to study the probability that the output of the network deviates from its nominal value by a given threshold. We derive a simple concentration inequality for the propagation of the input uncertainty through the network using the Cramer-Chernoff method and estimates of the local variation of the neural network mapping computed at the training points. We further discuss and exploit the resulting condition on the network to regularize the loss function during training. Finally, we assess the proposed tail probability estimate empirically on three public regression datasets and show that the observed robustness is very well estimated by the proposed method.

## 1 Introduction

Deep neural networks have proven to be very effective in practice to perform highly complex learning tasks [11]. Due to this success, they have gained a great deal of attention these past few years and they have been applied widely. However, they also have been found to be very sensitive to data uncertainties [9, 23] to the point that a whole research community is now addressing the so-called network attacks in order to study and design input noise that can fool the network decision. Attacks can be random, when data is corrupted by some random noise or adversarial, when the noise is specifically designed to alter the network output [23]. Even though both types of attacks are related since they are both addressing the robustness of the network, we will only focus in this article on the random case. Most data are usually uncertain, either because the data are related to naturally noisy phenomenon and we only have access to some of its statistics or because assessing devices to do not have sufficient accuracy to record precisely the data. In this study, we will therefore assume that the network input data is corrupted by some additive bounded random noise.

Robustness to bounded input perturbations has been analyzed in the past few years. Most people have addressed the problem through the use of regularization techniques [12, 19, 26, 10]. The main idea is to consider the neural network as a Lipschitz map between the input and output data. The Lipschitz constant of the network is then estimated or upper bounded by the norm of the layer-by-layer weights product. This estimates the expansion or contraction capability of the network and is then used to regularize the loss during training. Often, there is a price to pay: the expressivity of the network may be reduced, especially if the weights are too constrained or constrained layer by layer instead of constrained across layer [7]. Such strategies are enforcing robustness but do not provide guarantees or estimates on the level of robustness that has been achieved. In the case of adversarial perturbation, some authors have proposed methods for certifying robustness [15, 4]. Recently, a probabilistic approach has also

been proposed in the case of random noise for convolutional neural networks [27]. Pointing out that the threat of random noise may have been overlooked by the research community in favor of adversarial attacks, the authors have proposed probabilistic bounds based on the idea that the output of the network can be lower and upper bounded by two linear functions. The work proposed here is along the same line but distinct in several aspects. It combines upper bounds on tail probabilities calculated by deriving a specific Cramer-Chernoff concentration inequality for the propagation of uncertainty through the network with a network sensitivity estimate based on a network gradient calculation with respect to the inputs. The network gradient is computed by automatic differentiation and estimates the local variation of the output with respect to the input of the network. The estimation is carried out and averaged over the complete training set. A maximum component-wise gradient variation is also calculated in order to give probabilistic certificates rather than estimates. The certificates can be used in place of estimates whenever guaranteed upper bounds are needed, however they are often not as accurate since they are based on variation bounds rather than averages. We then discuss the use of the derived bounds and estimate in the design of deep robust neural networks and design experiments in order to assess the robustness probabilistic estimates for various regularization strategies.

The article is organized as follows: Section 2 provides the specific neural network concentration inequality using the Cramer-Chernoff method and the calculation of the network gradient estimate. Section 3 deals with training of the neural network and its regularization issues to increase its robustness. Section 4 provides the results of an empirical evaluation of the neural network robustness for three public datasets. Finally, Section 5 concludes the article.

## 2 Probabilistic certificates of robustness

Consider feed-forward fully connected neural networks that we represent as a successive composition of linear weighted combination of functions such that  $x^l = f^l((W^l)^\top x^{l-1} + b^l)$  for  $l = 1, \dots, L$ , where  $x^{l-1} \in \mathbb{R}^{n_{l-1}}$  is the input of the  $l$ -th layer, the function  $f^l$  is the  $L_f$ -Lipschitz continuous activation function at layer  $l$ , and  $W^l \in \mathbb{R}^{n_{l-1} \times n_l}$  and  $b^l \in \mathbb{R}^{n_l}$  are the weight matrix and bias vector between layer  $l-1$  and  $l$  that define our model parameter  $\theta = \{W^l, b^l\}_{l=1}^L$  that we want to estimate during training. The network can be seen as the mapping  $g_\theta : x^0 \rightarrow g_\theta(x^0) = x^L$ . The training phase of the network can be written as the minimization of the empirical loss  $\mathcal{L}(x, y, \theta) = \frac{1}{n} \sum_{i=1}^n l_\theta(g_\theta(x_i), y_i)$  where  $l_\theta$  is a measure of discrepancy between the network output and the desired output.

Assume now that we only have access to noisy observations  $x_i$  of the input sample. However, we know that these observations are drawn from a distribution  $D$  with finite support. We first consider the special case where the functions  $f_l$  are linear or piece-wise linear (this includes the case of ReLU activation functions) and then extend the analysis to more general functions. We start by considering the one layer NN setting to further extend the idea to several layers.

### 2.1 The single layer architecture

In this section, we consider the simple case where the outputs of the network  $y = x^L$  (with  $L = 1$  in this case) depends linearly of the inputs  $x = x^0 \in \mathbb{R}^n$  as follows:  $w^\top x + b$  where  $w = W^1$  is the single layer weights vector and  $b = b^1$  is the layer bias.

We assume that our input observations are corrupted by some additive noise  $\epsilon$  such that  $\epsilon \sim D$  and  $\forall i = 1, \dots, n$ , we have  $\epsilon_i \in [-\gamma, \gamma]$  with  $\gamma < +\infty$ . Our objective is to ensure the following property:

$$\mathbb{P}_{\epsilon \sim D} (\|y - y_\epsilon\| \leq \Gamma) \geq 1 - \alpha \quad (1)$$

where  $y_\epsilon = w^\top(x + \epsilon) + b$ ,  $\Gamma$  is the allowed output uncertainty and  $1 - \alpha$  is some predefined level of confidence.

In the following proposition, we give a condition on  $w$  that will ensure, with probability greater than  $1 - \alpha$ , that the output uncertainty remains below  $\Gamma$ .

**Proposition 2.1** *If the network inputs are subject to an additive uncertainty  $\epsilon$  where  $\forall i = 1, \dots, n$ ,  $\epsilon_i \sim D$  and  $\text{supp}(D) \in [-\gamma, \gamma]$  ( $\gamma < +\infty$ ), then for a given  $\alpha \in [0, 1]$  and*

a given output uncertainty level  $\Gamma$ , the following condition holds:

If the layer weights vector  $w$  satisfies  $\|ww^\top\|_F \leq \frac{\Gamma^2}{\gamma^2 \sqrt{2 \log(1/\alpha)}}$ , then  $\mathbb{P}_{\epsilon \sim D}(\|y - y_\epsilon\| \leq \Gamma) \geq 1 - \alpha$ .

**Proof:** We start by observing that in general we have

$$\mathbb{P}_{\epsilon \sim D}(\|y - y^\epsilon\| \leq \Gamma) \geq \mathbb{P}_{\epsilon \sim D}\left(\text{tr}\left(w^\top \epsilon (w^\top \epsilon)^\top\right) \leq \Gamma^2\right), \quad (2)$$

where  $\text{tr}(A)$  defines the trace of a matrix  $A$ . Note that in the particular case where all activation functions are linear, equality is achieved.

Therefore inequality (1) may be satisfied by ensuring that

$$\mathbb{P}_{\epsilon \sim D}\left(\text{tr}(w^\top \epsilon (w^\top \epsilon)^\top) \geq \Gamma^2\right) = \mathbb{P}_{\epsilon \sim D}\left(\text{tr}(\epsilon \epsilon^\top w w^\top) \geq \Gamma^2\right) \leq \alpha.$$

We now state and prove the following Lemma which provides a simple concentration inequality for the above probability.

**Lemma 2.2** For any random matrix  $M \in \mathbb{R}^{n \times n}$  of the form  $M = vv^\top$  where  $v$  is a random vector such that for all  $i$  in  $\{1, \dots, n\}$ ,  $v_i$  are all independent, have finite support included in  $[-\delta, \delta]$  and  $\mathbb{E}(v_i) = 0$ , we have

$$\forall Q \in \mathbb{R}^{n \times n}, \forall t > 0, \quad \mathbb{P}(\text{tr}(MQ) \geq t) \leq e^{-\frac{t^2}{2\beta^2 \delta^4 \|Q\|_F^2}}. \quad (3)$$

where  $\beta = \min\{\rho > 0 \text{ s.t. } \frac{(\rho z)^2}{2} - \log(\cosh(z)) \geq 0\}$ .

**Proof:** The proof is based on the Cramer-Chernoff method [5] to bound the tail probability of the random variable  $\text{tr}(MQ)$ . Applying Markov inequality to the left hand side of (3), we have:

$$\mathbb{P}(\text{tr}(MQ) \geq t) \leq \frac{\mathbb{E}(\text{tr}(MQ))}{t}$$

and for any  $p \in \mathbb{R}^+$ , since  $m_{ij} = v_i v_j = m_{ji}$ ,

$$\begin{aligned} \mathbb{P}(\text{tr}(MQ) \geq t) &\leq e^{-pt} \mathbb{E}\left(e^{p \text{tr}(MQ)}\right) \\ &\leq e^{-pt} \mathbb{E}\left(e^{p \sum_{i,j=1}^n m_{ij} q_{ji}}\right) \\ &\leq e^{-pt} \mathbb{E}\left(\left(\prod_{i=1}^n e^{p m_{ii} q_{ii}}\right) \left(\prod_{i,j=1:i < j}^n e^{2p m_{ij} q_{ji}}\right)\right). \end{aligned} \quad (4)$$

Observe now that since for all  $i \neq j$ , we have that  $\text{cov}(m_{ii}, m_{jj}) = 0$  and since  $\mathbb{E}(v_i) = \mathbb{E}(v_j) = 0$  that

$$\text{cov}(m_{ii}, m_{ij}) = \text{cov}(v_i v_i, v_i v_j) = 0.$$

Therefore, from (4), we have

$$\mathbb{P}(\text{tr}(MQ) \geq t) \leq e^{-pt} \prod_{i,j=1}^n \mathbb{E}(e^{p m_{ij} q_{ji}}).$$

Let now  $\psi_{\text{tr}(MQ)}$  be the moment generating function of  $\text{tr}(MQ)$ , its Cramer transform obtained by Fenchel-Legendre duality is

$$\psi_{\text{tr}(MQ)}^*(t) = \sup_{p \geq 0} (pt - \psi_{\text{tr}(MQ)}(z)).$$

Note that  $m_{ij} \in [-\delta^2, \delta^2]$  and that by convexity of the exponential function, we can bound the moment generating function of  $m_{ij}$  as follows:

$$\forall z \in \mathbb{R}, \quad \mathbb{E}(e^{z m_{ij}}) \leq \frac{1}{2} e^{-z \delta^2} + \frac{1}{2} e^{z \delta^2} = \cosh(z \delta^2).$$

Therefore, around zero, we can write

$$\log(\mathbb{E}(z m_{ij})) \leq \log(\cosh(z \delta^2)) \leq \frac{(\beta z \delta^2)^2}{2}$$

where  $\beta$  is a coefficient in  $(0, 1]$  that tightens the bound as much as possible and can be defined as  $\beta = \min\{\rho > 0 \text{ s.t. } \frac{(\rho z)^2}{2} - \log(\cosh(z)) \geq 0\}$ . Replacing  $z$  by  $pq_{ji}$  with  $p > 0$ , we can derive the following upper bound for the Cramer transform of  $\psi_{\text{tr}(MQ)}^*$ :

$$\forall t \in \mathbb{R}, \quad \psi_{\text{tr}(MQ)}^*(t) \leq \min_{p>0} \left\{ -pt + \sum_{i,j=1}^n \frac{(pq_{ji}\beta\delta^2)^2}{2} \right\}.$$

The minimum in the right hand side of the expression above is reached at  $p^* = \frac{t}{\beta^2\delta^4\|Q\|_F^2}$  and therefore, applying Cramer inequality, we finally get

$$\log(\mathbb{P}(\text{tr}(MQ) \geq t)) \leq -\frac{t^2}{2\beta^2\delta^4\|Q\|_F^2},$$

which completes the proof of the lemma.  $\square$

Note that  $\forall(i, j) \in \mathbb{R}^{n \times n}$ , we have  $(\epsilon\epsilon^\top)_{ij} \in [-\gamma^2, \gamma^2]$ . Hence, applying Lemma 2.2 to bound  $\mathbb{P}_{\epsilon \sim D}(\text{tr}(\epsilon\epsilon^\top ww^\top) \geq \Gamma^2)$  will lead to

$$\mathbb{P}_{\epsilon \sim D}(\text{tr}(\epsilon\epsilon^\top ww^\top) \geq \Gamma^2) \leq e^{-\frac{\Gamma^4}{2\beta^2\gamma^4\|ww^\top\|_F^2}} \leq e^{-\frac{\Gamma^4}{2\gamma^4\|ww^\top\|_F^2}},$$

since  $\beta \in (0, 1]$ . In order to remain below the level  $\alpha$ , we then need that

$$-\frac{\Gamma^4}{2\gamma^4\|ww^\top\|_F^2} \leq \log(\alpha). \quad (5)$$

This can also be written as  $\|ww^\top\| \leq \frac{\Gamma^2}{\gamma^2\sqrt{2\log(1/\alpha)}}$ , proving Proposition 2.1.  $\square$

Note that in (5), one can keep the tightening coefficient  $\beta$  to get a sharper bound whenever needed and write  $\|ww^\top\| \leq \frac{\Gamma^2}{\beta\gamma^2\sqrt{2\log(1/\alpha)}}$ . The value of the coefficient of the matrix  $\beta$  can be estimated numerically.

## 2.2 Extension to deeper architectures

In this section, we address the case where the network is composed of several layers  $l = 1, \dots, L$  with linear or piece-wise linear activation functions. Property (1) should now relate to the output of the last layer as follows:

$$\mathbb{P}_{\epsilon \sim D}(\|x^L - x_\epsilon^L\| \leq \Gamma) \geq 1 - \alpha \quad (6)$$

where  $x_\epsilon^L$  is the output of the layer  $L$  when a noisy input  $x_\epsilon^0 = x^0 + \epsilon$  is propagated through the network and  $\epsilon$  is the additive input noise such that  $\epsilon \sim D$  and  $\forall i = 1, \dots, n$ , we have  $\epsilon_i \in [-\gamma, \gamma]$  ( $\gamma < +\infty$ ). With this setting, we can now state and prove the following:

**Proposition 2.3** *If the network inputs are subject to an additive uncertainty  $\epsilon$  where  $\forall i = 1, \dots, n$ ,  $\epsilon_i \sim D$  and  $\text{supp}(D) \in [-\gamma, \gamma]$  ( $\gamma < +\infty$ ), then for a given  $\alpha \in [0, 1]$  and a given output uncertainty level  $\Gamma$ , the following condition holds:*

*If the layer weights vector  $w$  satisfies  $\left\| \left( \prod_{l=1}^L W^l \right) \left( \prod_{l=1}^L W^l \right)^\top \right\|_F \leq \frac{\Gamma^2}{\gamma^2\sqrt{2\log(1/\alpha)}}$*

*then  $\mathbb{P}_{\epsilon \sim D}(\|x^L - x_\epsilon^L\| \leq \Gamma) \geq 1 - \alpha$ .*

**Proof:** Propagating forward the input uncertainties through the network, we can write:

$$x^L - x_\epsilon^L = (W^L)^\top \epsilon^{L-1} = (W^L)^\top (W^{L-1})^\top \epsilon^{L-2} \dots (W^1)^\top \epsilon = \left( \prod_{l=1}^L W^l \right)^\top \epsilon$$

where  $\epsilon^l$  is the propagated noise from layer 1 to layer  $l$ . Furthermore, we have

$$\begin{aligned} \left\| \left( \prod_{l=1}^L W^l \right)^\top \epsilon \right\|^2 &= \text{tr} \left( \left( \prod_{l=1}^L W^l \epsilon \right) \left( \prod_{l=1}^L W^l \epsilon \right)^\top \right) \\ &= \text{tr} \left( \left( \prod_{l=1}^L W^l \right) \epsilon \epsilon^\top \left( \prod_{l=1}^L W^l \right)^\top \right) \\ &= \text{tr} \left( \epsilon \epsilon^\top \left( \prod_{l=1}^L W^l \right) \left( \prod_{l=1}^L W^l \right)^\top \right). \end{aligned}$$

Therefore, applying again Lemma 2.2 to upper bound the probability in (6), the following condition on the layer weights matrices to ensure property (6) is directly obtained:

$$\left\| \left( \prod_{l=1}^L W^l \right) \left( \prod_{l=1}^L W^l \right)^\top \right\|_F \leq \frac{\Gamma^2}{\gamma^2 \sqrt{2 \log(1/\alpha)}}. \quad (7)$$

□

### 2.3 Improving the bound by taking the network nonlinearity into account

Remember that the bound derived above relies on the fact that we have considered the linear upper bound of the neural network response and therefore from inequality (2) applied to the multilayers case, we have

$$\mathbb{P}_{\epsilon \sim D} \left( \|x^L - x_\epsilon^L\| \leq \Gamma \right) \geq \mathbb{P}_{\epsilon \sim D} \left[ \text{tr} \left( \epsilon \epsilon^\top \left( \prod_{l=1}^L W^l \right) \left( \prod_{l=1}^L W^l \right)^\top \right) \leq \Gamma^2 \right] \quad (8)$$

The Chernoff bound proposed above may be tight with respect to the right hand side of the above inequality, however, with respect to the left hand side, (8) is not tight in general as the deep complex layer structure of the neural network generates a highly non linear behavior. The linear upper bound is often of poor quality. To address this issue, we propose an alternative to estimate the variation of the neural network response. In the following, we will consider that the network output is 1-dimensional. The following developments also generalize to the multi-dimensional output case by using differential calculus for multi-dimensional valued mappings but at the unnecessary expense of clarity. Since  $\|\epsilon\|$  may be considered small with respect to the magnitude of the network inputs, we are interested in the local behavior of the network output that we will approximate as follows:

$$x_\epsilon^L - x^L = F(x + \epsilon) - F(x) \simeq \nabla_x F(x)^\top \epsilon \quad (9)$$

where  $F(x) = f^L((W^L)^\top f^{L-1}((W^{L-1})^\top f^{L-2}(\dots f^0((W^0)^\top x + b^0)) + b^{L-2}) + b^{L-1})$  and  $\nabla_x F$  is the gradient of  $F$  with respect to the network input  $x$ .

The linear approximation (9) is only local but its advantage is that it can easily be evaluated at many  $x$  values. Indeed, while visiting all input vector  $x$  during training, this information is usually available at a very low extra computational cost through automatic differentiation in most training computer algorithms and packages for neural networks (such as TensorFlow [1] and Pytorch [21]). Therefore, from local estimates at various training points  $x_i^0$  for  $i = 1, \dots, n$ , we calculate two  $n_0$ -dimensional vectors of network variation estimates  $\widehat{\nabla_x F}$  and  $\overline{\nabla_x F}$  defined as follows:

$$\left( \widehat{\nabla_x F} \right)_k = \text{sign} \left( \left( \nabla_x F(x_{i_k}^0) \right)_k \right) \times v_k \quad \overline{\nabla_x F} = \frac{1}{n} \sum_{i=1}^n \nabla_x F(x_i^0) \quad (10)$$

where

$$(v_k, i_k) = \left( \max_{i \in \{1, \dots, n\}} \left| \left( \nabla_x F(x_i^0) \right)_k \right|, \arg \max_{i \in \{1, \dots, n\}} \left| \left( \nabla_x F(x_i^0) \right)_k \right| \right).$$

The quantity  $\widehat{\nabla_x F}$  accounts for the maximum variation of the network response in every component direction of the network input encountered during the training and gives therefore a tighter linear upper bound than  $\left(\prod_{l=1}^L W^l\right)$  when input data are part of the training set. The quantity  $\overline{\nabla_x F}$  does not provide a linear upper bound but estimates an average linear behavior of the network response that be used in practice to estimate the required  $\Gamma$  value to reduce  $\mathbb{P}_{\epsilon \sim D} (\|x^L - x_\epsilon^L\| \geq \Gamma)$  below  $\alpha$ . The larger is the training dataset, the higher is the quality of these estimates. Replacing  $\left(\prod_{l=1}^L W^l\right)$  by these quantities in (7), we derive the following robustness bound for the network:

$$\left\| \widehat{\nabla_x F} \widehat{\nabla_x F}^\top \right\|_F \lesssim \frac{\Gamma^2}{\gamma^2 \sqrt{2 \log(1/\alpha)}}. \quad (11)$$

and the following estimate of  $\Gamma$  to achieve a robustness confidence level of  $1 - \alpha$ :

$$\Gamma \gtrsim \gamma \left( 2 \log(1/\alpha) \|\overline{\nabla_x F} \overline{\nabla_x F}^\top\|_F^2 \right)^{\frac{1}{4}}. \quad (12)$$

## 2.4 Discussion on the bound

The bound given in (11) and (12) are interesting in practice in several ways. First, they can be used as a network probabilistic robustness estimates. Indeed, assume that the magnitude of the input uncertainty  $\gamma$  is known and a level of uncertainty  $\Gamma$  is required for a practical application. Then set a level of confidence  $1 - \alpha$ . After training, if the network weights satisfy the bound (11), then it implies that the output uncertainty will be below the required level  $\Gamma$  with a propability around  $1 - \alpha$ . Of course, obtaining strict guarantees of robustness would be even more interesting. However, getting a strict bound inequality in (11) and (12) would require a strict upper bound on the network local variation. An available upper bound is the global linear upper bound that we have calculated in Section 2.2 but at the expense of bound tightness. For this reason, instead of computing upper bounds that are loose, we propose estimates of bounds that do not provide strict certificate of robustness but are tighter to what is observed in practice as we will see in Section 4.

Alternatively, assume now that you know  $\gamma$  and after training you observe the quantity  $\|\overline{\nabla_x F} \overline{\nabla_x F}^\top\|_F$ . For a level of confidence  $1 - \alpha$ , (12) gives an estimate of the maximum variation  $\Gamma$  of the output that you might expect. Therefore, for a specific application, the calculated  $\Gamma$  provides an additional performance criterion for the trained network. In addition to validation accuracy and generalization performance, the  $\Gamma$  value gives an estimate of the sensitivity of the network to input uncertainty. The larger is  $\Gamma$ , the more sensitive to uncertainty is the network since, for the same level of  $\alpha$  and  $\gamma$ , one has to expect that the output should deviate (in norm) from a nominal value by  $\Gamma$ . Finally, (11) and (7) emphasize also the relationship between the robustness of the network and the weights values achieved after training. One idea is to use this knowledge to drive the training process to regions where the quantity on the left hand side of these bounds are small in order to design networks that are intrinsically robust. This is the objective of the next section.

## 3 Controlling the bound during training

In this section, we are interesting in exploiting the bounds derived above during the process of training the neural network. The main idea would be to ensure that optimal weights after training are satisfying the bound constraint (11). Naturally, this could be formulated as a constrained optimization training problem. Stochastic projected gradient techniques [18, 16] could be used to solve such a problem. However, in the general case, the projection operator for such constraint is not simple and would require important computational effort. Therefore, instead of ensuring the constraint, we propose to regularize the loss function during training by adding a penalization term as follows:

$$\min_{\theta=(W,b)} \frac{1}{\lambda} \mathcal{L}(x, y, \theta) + \|Q_W\|_F^2$$

where  $\lambda$  is a positive parameter,  $\mathcal{L}$  is a loss function (mean squared error for example) and  $\|Q_W\|_F$  is the Frobenius norm of a matrix  $Q_W$  that could be chosen as  $Q_W = \widehat{\nabla_x F} \widehat{\nabla_x F}^\top$ ,  $Q_W = \overline{\nabla_x F} \overline{\nabla_x F}^\top$  or  $Q_W = \left(\prod_{l=1}^L W^l\right) \left(\prod_{l=1}^L W^l\right)^\top$ , depending on which bound from above we want to exploit. Regularization is a common practice in machine learning [3, 11] and is usually proposed to avoid overfitting and increase model generalization. The connection between generalization and robustness to input uncertainty in machine learning models has been established in several studies [28, 22]. Intuitively, the  $\|Q_W\|_F^2$  regularization term acts as a special weight contraction and it is natural to consider alternative possibilities to reduce the magnitude of the network weights. One alternative is the squared spectral norm (largest eigenvalue) of  $Q_W$  that would also account for the maximum absolute contraction of a vector when multiplied by  $Q_W$ . Finally, in [7], the product  $\prod_{l=1}^L \|W^l\|$  which is an upper bound of the Lipschitz constant of the network has also been proposed as regularization that promotes robustness. It accounts for the overall Lipschitz regularity of the network and acts also as an overall control on the contraction power of the network by coupling layers and allowing some weights to grow for some layers as long as in other layers others weights are getting smaller to compensate.

When  $Q_W = \left(\prod_{l=1}^L W^l\right) \left(\prod_{l=1}^L W^l\right)^\top$ , its Frobenius norm and the Lipschitz constant gradient can be explicitly derived and integrated into the backpropagation scheme and chain rule of gradients in order to optimize the augmented loss during the training phase. However, for the spectral norm, approximation methods are necessary and gradient will have to be computed using numerical differentiation techniques. Among available approximation methods, the power iteration algorithm [2], or preferably the Lanczos algorithm [17] since  $Q_W$  is symmetric, is well suited for the purpose. Note that there is no real need to approximate  $\lambda_{max}(Q_W)$  (the largest eigenvalue of  $Q_W$ ) with great accuracy as it is only used to as a regularization function to guide the optimization process towards optimal regions where the spectral norm is reduced. Therefore, an alternative approximation technique is to use an upper bound of  $\lambda_{max}(Q_W)$ . As  $Q_W$  is positive definite, we propose to use the Dembo’s upper bound [8] defined as follows: Let  $A_n \in \mathbb{R}^{n \times n}$  be an Hermitian positive definite matrix and let  $\lambda^{(n)}_1 \leq \dots \leq \lambda^{(n)}_n$  be the eigenvalues of  $A_n$ . The matrix  $A_n$  can be written as

$$A_n = \begin{pmatrix} A_{n-1} & b \\ b^H & c \end{pmatrix}$$

where  $b^H$  denotes the Hermitian transpose and the largest eigenvalue of  $A_n$  satisfies

$$\lambda_n^{(n)} \leq \frac{c + \lambda_n^{(n-1)}}{2} + \sqrt{\frac{(c - \lambda_n^{(n-1)})^2}{4} + b^H b}$$

In the next section, we propose to carry out experiments with these various regularization strategies and evaluate their respective impact on the robustness properties of the network.

## 4 Experiments

In order to assess the quality of the calculated bound, experiments are conducted on public datasets. We focus on deep neural network regression tasks (linear output activation) and the **BOSTON** [13], **DIABETES** [25] and **CALIFORNIA** [20] datasets. The neural network and its training and testing are implemented in the `python` [24] environment using the `keras` [6] library and `Tensorflow` [1] backend. The neural network architecture is composed of 4 dense hidden layers with 50 ReLU activations neuronal units and one dense linear output layer. All results that are presented below are average results from 10 independent runs that are carried out after random shuffling and random splitting of datasets. All dataset samples are scaled so that they lie in  $[-1, 1]$ . All neural network training procedures are executed with the `ADAM` stochastic optimization algorithm with default parameters as given in [14]. Additional details about the datasets dimensions and training parameters are given in Table 1.



Table 1: Dataset information and experimental setup

Dataset	# inputs	# samples	test/train ratio	batch size	# epochs	learning rate
BOSTON	13	606	0.2	50	100	0.001
DIABETES	10	442	0.2	200	30	0.001
CALIFORNIA	8	20640	0.4	600	30	0.001

All comparison results provided below are referring to the following three methods as described in section 3:

- **no reg**: training procedure with mean squared error loss without regularization
- **Lipschitz reg**: training procedure with mean squared error loss with the  $\prod_{l=1}^L \|W^l\|$  regularizer as described in [7].
- **Gradient reg**: training procedure with mean squared error loss with the  $\|Q_W\|^2$  regularizer as described above and where  $Q_W = \widehat{\nabla_x F} \widehat{\nabla_x F}^\top$ .
- **MaxEig reg**: training procedure with mean squared error loss with the  $\lambda_{max}(Q_W)^2$  regularizer.

In order to estimate the probability  $\mathbb{P}_{\epsilon \sim D}(\|y - y_\epsilon\| \leq \Gamma)$ , the following procedure is applied. For each test sample from the validation set, we generate random vectors  $\epsilon_j$  with  $j \in \{1, \dots, 10\}$  and calculate the following probability estimate:

$$\frac{1}{10 \times T} \sum_{i=1}^T \sum_{j=1}^{10} \mathbf{1}_{\{\|y^{(i)} - y_{\epsilon_j^{(i)}}\| \leq \Gamma\}}(\epsilon_j) \quad (13)$$

where  $T$  is the number of samples in the testing set,  $y^{(i)}$  is the desired output for the  $i$ -th testing sample and  $y_{\epsilon_j^{(i)}}$  is the output of the network calculated via a forward pass through the network for the input vector  $x^{(i)} + \epsilon_j$ . In all experiments, we have  $supp(D) = [-\gamma, \gamma]$  with various levels of  $\gamma$ . In the figures described below,  $\gamma$  is referred as **gamma** and  $\Gamma$  as **Gamma**.

Figure 1(right) reports these observed probabilities together with the estimated tail probabilities given by

$$e^{-\frac{\Gamma^4}{2\gamma^4 \|\widehat{\nabla_x F} \widehat{\nabla_x F}^\top\|_F^2}}$$

for various values of  $\Gamma$ , while Figure 1(left) reports the corresponding mean validation error achieved during the training process. The probability level  $1 - \alpha$  (with  $\alpha = 0.05$ ) is also marked with a blue dashed line on each plot on the right. Figure 2 provides further details about the magnitude of the norm of the network gradient  $\|\widehat{\nabla_x F}\|$  and  $\lambda_{max}(Q_W)^2$  (re-scaled by a factor 10 to ease the reading of the plot) for each dataset and the four regularization strategies. Finally, Figure 3 provides, for each dataset and each regularization strategies a comparison of the  $\Gamma$  values achieved to reach a  $1 - \alpha$  probability level. Three values are reported each time,  $\Gamma(max) = \left(2 \log(1/\alpha) \|\widehat{\nabla_x F} \widehat{\nabla_x F}^\top\|_F^2\right)^{\frac{1}{4}}$ ,  $\Gamma(mean) = \left(2 \log(1/\alpha) \|\widehat{\nabla_x F} \widehat{\nabla_x F}^\top\|_F^2\right)^{\frac{1}{4}}$  and the  $\Gamma$  value observed so that the probability given in (13) reaches a level  $1 - \alpha$ .

We see in Figure 1 that, for the three datasets and for a probability of 0.95, the calculated  $\Gamma$  value ( $x$  axis) obtained by the expression of the exponential tail probability, provides a very good estimate of the  $\Gamma$  value given by the observed probability (probability that the output deviates from its nominal value by more than  $\Gamma$ ). This validates experimentally, at least for these datasets, the relevance of the estimate given in (12). For the **BOSTON** dataset, all regularizing strategies give similar  $\Gamma$  values whereas for the **DIABETES** and **CALIFORNIA**, the  $\Gamma$  values are more sensitive to the type of regularization employed. However, surprisingly, no general rule can be given from these results. It is difficult to

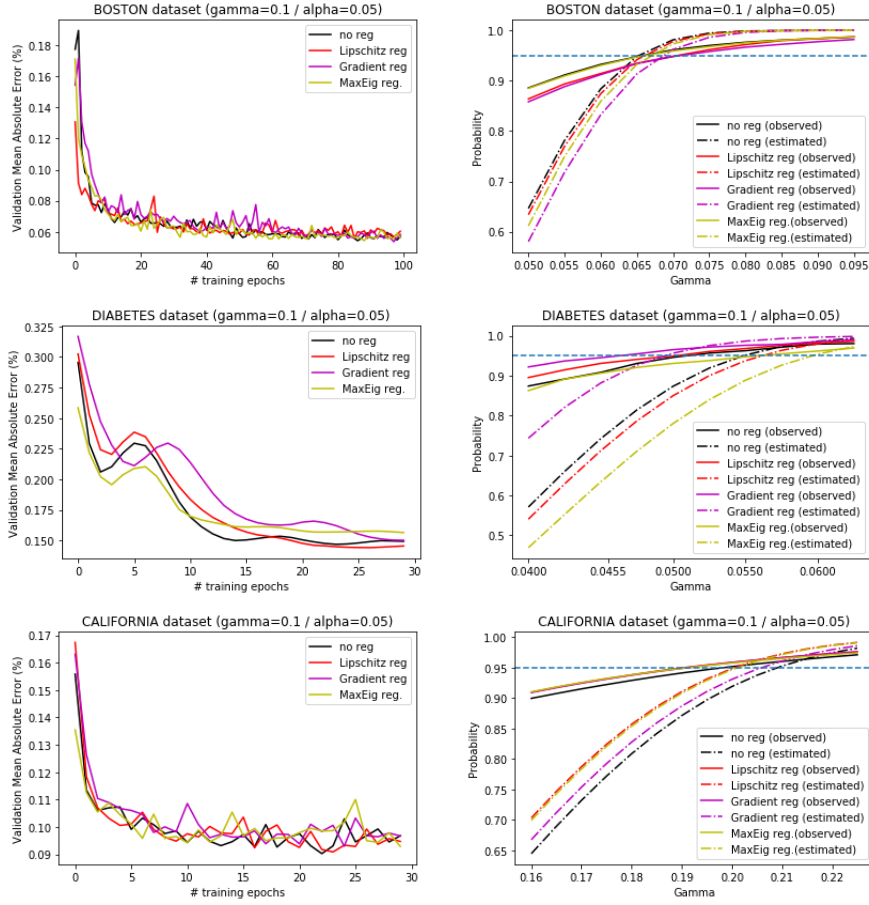


Figure 1: Mean absolute validation error profiles during training (left) & Comparison of  $\mathbb{P}_{\epsilon \sim D} (\|y - y_\epsilon\| \leq \Gamma)$  and exponential tail bound for various levels of  $\Gamma$  (right)

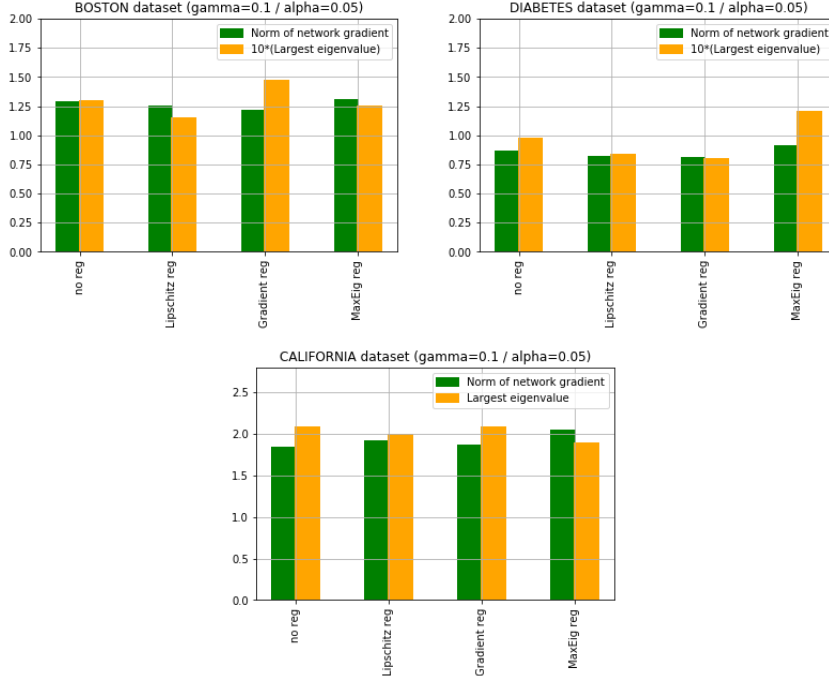


Figure 2: Norm of network gradient  $\|\widehat{\nabla_x F}\|$  and Estimate of  $\lambda_{max}(Q_W)$  after training

say which regularizer performs best. It is dataset dependent. The left hand side plots of Figure 1, representing the validation mean absolute error, show that similar training performance were achieved by the four regularizing methods and do not provide further explanation of this phenomenon. We believe, without providing any evidence of it, that the high nonlinearity of the neural network error surface may explain it. Indeed, after training, the optimization algorithm has reached a local minimum where the loss value may not have decreased sufficiently to really express the regularization power of the regularizer. This depends on the geometry of the error surface that is greatly dependent on the input data.

On Figure 2, the norm of the network gradient tends to be slightly smaller for the **Gradient reg** strategy. This would confirm that regularizing by the network gradient would help in achieving better robustness. Additionally, the figure also shows that the maximum eigenvalue regularization is not correlated to the network gradient norm and may not be a suitable alternative for robustness purposes. The  $\Gamma$  value comparison in Figure 3 confirms that the  $\Gamma$  estimates calculated by the proposed method are very closed to the observed values. This is true for all datasets and regularizing strategies. Furthermore, the  $\Gamma$  upper bound values ( $\Gamma(max)$ ), are loose as expected in Section 2.3 but provide certificates for robustness. These certificates follow the same pattern as the norm of the network gradient in Figure 2, which was also expected since their expression in (11) are directly dependent. Therefore, as for the network gradient, we observed that these certificates are better (tighter but still quite loose) for the network gradient regularizing strategy.

## 5 Conclusions

In this study, we have proposed analytical probabilistic estimates (and certificates) for deep dense neural networks. The idea combines tail probability bound calculation using the Cramer-Chernoff scheme and the estimation of the network local variation.

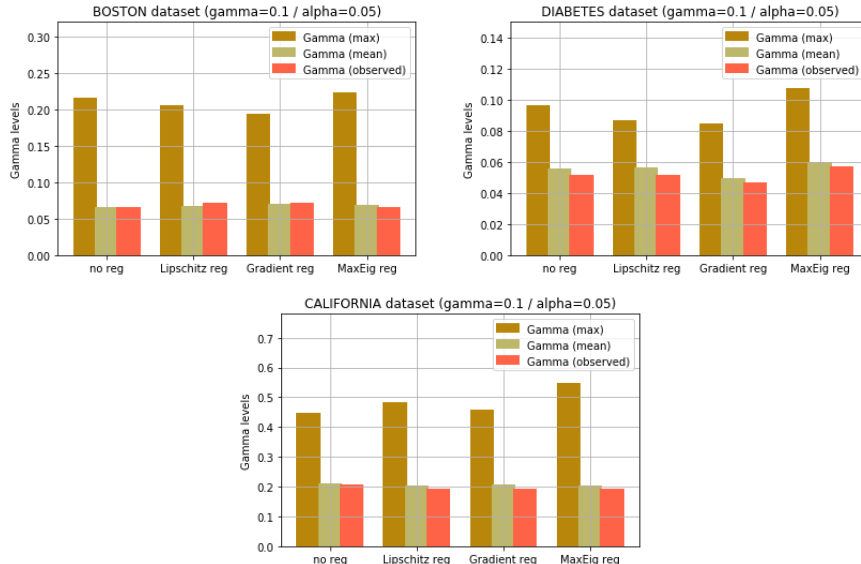


Figure 3: Comparison of calculated  $\Gamma(max)$  and  $\Gamma(mean)$  values for 0.95 confidence level using  $\widehat{\nabla}_x F$  and  $\bar{\nabla}_x F$  respectively with the  $\Gamma(observed)$  value observed so that the probability given in (13) reaches a level 0.95

The network gradient computation is using the automatic differentiation procedure available in many neural network training packages and carried out only at the training samples which does not require much extra computational cost. Experiments with this method has been conducted on public datasets and has shown that the robustness estimates are very good compared to the observed network robustness. Further analysis on these datasets show that the quality of the estimates is not really impacted by the regularization strategy, however, the network gradient regularization tends to generate slightly more robust network architectures.

## Acknowledgements

Our work has benefitted from the AI Interdisciplinary Institute ANITI. ANITI is funded by the French "Investing for the Future - PIA3" program under the Grant agreement # ANR-19-PI3A-0004.

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