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Article

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Abstract: In this work, we show a construction of a Hessian foliation on a statistical manifold. Our results provide conditions, under which the compacts hessian leaves (up to diffeomorphism) are quotients of homogeneous convex cones.

1. Introduction

The Hessian structures play important roles in Information geometry. In [1], Shima observes that Fisher information of all classical used statistical models are Hessian metrics. As Amari did he also proved that any locally flat statistical manifold is a Hessian manifold. These structures arised from works of many mathematicians (e.g. J.-L. Koszul, Y. Matsushima, A. Nijenhuis, E. B. Vinberg) in their attempt to solve the Gerstenhaber conjecture in the category of locally flat manifold.The geometry of locally flat hyperbolic manifold is also named Koszul geometry (see [2]),the conjecture of Gerstenhaber say that every restricted theory of deformation generates its proper theory of cohomology.This conjecture has been solved in 2006 (see [3]) by Michel Boyom. In[4][5] he also realize the homological version of the Hessian geometry. The aim of this paper is to study hyperbolic foliations in statistical manifolds . Here a hyperbolic foliation is a foliation whose leaves support Koszul geometry.The paper is organised as it follows. In section is this introduction. Section 2 is devoted to basics notions which are used. Section 3 is devoted to useful notions in the category of locally flat manifolds and their KV-cohomology. Section 4 is devoted to Hessian equation of Koszul connection in tangent bundle of statistical manifolds. In section 5 we focus on a cononical pair of locally flat foliations in statistical manifolds ,and we discuss conditions under which these foliations are hyperbolic .

2. Statistical manifold

2.1. Koszul connections

We recall in this subsection some basics notions. All differentiable manifold we deal with are paracompact. The class of differentiability is C^∞. For a manifold M the real algebra of differentiable real valued function is denote by $C^\infty(M)$. The $C^\infty(M)$-module of differentiable vectors fields is denoted by $\mathcal{X}(M)$.

Definition 1. A Koszul connection on M is a $\mathbb{R}$-bilinear product

$$(X,Y) \in \mathcal{X}(M)^2 \longrightarrow \nabla_X Y \in \mathcal{X}(M)$$

satisfying the following identities for $X, Y \in \mathcal{X}(M)$ and $f \in C^\infty(M)$:

$$\nabla_{fX} Y = f \nabla_X Y$$

$$\nabla_X(fY) = df(X)Y + f\nabla_X Y$$

Definition 2. Let ∇ be a koszul connection. Its torsion tensor T^∇ and curvature tensor R^∇ are defined as follows:

$$T^\nabla(X,Y) = \nabla_X Y - \nabla_Y X - [X,Y]$$

$$R^\nabla(X,Y) = \nabla_X \circ \nabla_Y - \nabla_Y \circ \nabla_X - \nabla_{[X,Y]}$$

2.2. Statistical Manifold

Definition 3. A statistical manifold is a triple (M, g, ∇) formed of a symmetric gauge structure (M, ∇) and a Riemannian structure (M, g) which are connected by the following relation:

$$(\nabla_X g)(Y, Z) = (\nabla_Y g)(X, Z).$$

It is easy verify that (M, g, ∇^*) is also a statistical manifold, where the symmetric connection ∇^* is defined by the following identity:

$$g(\nabla^*_X Y, Z) = X \cdot g(Y, Z) - g(Y, \nabla_X Y).$$

Example 1: Any Riemannian manifold (M, g, ∇^{LC}) where ∇^{LC} is the Levi-Civita connection of g, We will say that M is a trivial statistical manifold.

Example 2: Let (M, g, ∇, ∇^*) any statistical manifold, the family $(M, g, \nabla^{(\alpha)}, \nabla^{(-\alpha)})_{\alpha \in \mathbb{R}}$ where $\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*$ is a statistical manifold for $\forall \alpha \in \mathbb{R}$.

Example 3: Let (M, g) some Riemannian manifold and denote ∇^{LC} the Levi-Civita connection with respect to g, and let $E \in TM \setminus \{0\}$. The triplet (M, g, ∇, ∇^*) is a statistical manifold where $\nabla_X Y = \nabla^{LC}_X Y + g(X, E)g(Y, E)E$ and $\nabla^*_X Y = \nabla^{LC}_X Y - g(X, E)g(Y, E)E$.

3. KV-cohomology of Locally Flat Manifold

3.1. Locally Flat Manifold

Definition 4. A locally flat manifold is a pair (M, ∇) formed by a differentiable manifold M and a connection ∇ whose curvature tensor R^∇ and torsion tensor T^∇ vanish:

$$\nabla_X Y - \nabla_Y X - [X, Y] = 0$$

$$\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]} Z = 0,$$

$\forall X, Y, Z \in \Gamma(TM)$.

Definition 5. An affinely flat structure in a m-dimensional manifold M is defined by an complete atlas:

$$\mathcal{A} = \{(U_i, \Phi_i)\}$$

whose local chart changes $\Phi_i^{-1} \circ \Phi_j$ are the restrictions of affine transformations of the m-dimensional affine space $\mathbb{R}^m$.

Theorem 1. *There is a one-to-one correspondence between the category of affinely flat structures in M and the category of locally flat structures in M.*

*Example 5:*Let $\mathbb{R}^n = \{(x_1, ..., x_n)/x_i \in \mathbb{R}\}$ endowed with linear connection ∇^F defined by

$$\nabla^F \frac{\partial}{\partial x_i} = 0$$

$\forall i = 1, 2, .., n$ and $\forall X \in \mathcal{X}(M)$
Example 6: Let $\mathbb{R}^2$ and $\forall X = f\frac{\partial}{\partial x_1} + g\frac{\partial}{\partial x_2} \in \mathcal{X}(M)$, endowed by a linear connection defined by

$$\nabla \frac{\partial}{\partial x_1} = f\frac{\partial}{\partial x_1} \text{ and } \nabla \frac{\partial}{\partial x_2} = 0$$

$(\mathbb{R}^2, \nabla)$ is locally flat manifold.

To a locally flat manifold (M, ∇) we assigned the algebra $\mathcal{A}$ whose underlying vector space is the vectors fields, namely $\mathcal{X}(M)$, and the product $X \cdot Y$ is defined by the Koszul connection ∇ as:

$$X \cdot Y = \nabla_X Y$$

Definition 6. An algebra $\mathcal{B}$ is Koszul-Vinberg algebra (or KV in short) algebra if for any triple (a, b, c) in $\mathcal{B}^3$ the following identity is satisfy:

$$a(bc) - (ab)c - b(ac) + (ba)c = 0.$$

Proposition 1. *The algebra $A = (\mathcal{X}(M), \nabla)$ of a locally flat manifold (M, ∇) is a Koszul-Vinberg algebra.*

Proof. Since the curvature tensor vanishes, we get for any triple (X, Y, Z) of vector fields that:

$$\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) = \nabla_{[X,Y]} Z;$$

but ∇ is torsion-less, so the right hand side is:

$$\nabla_{\nabla_X Y} Z - \nabla_{\nabla_Y X} Z.$$

This yields:

$$\nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z - \nabla_Y(\nabla_X Z) + \nabla_{\nabla_Y X} Z = 0$$

or

$$X(YZ) - (XY)Z - Y(XZ) + (YX)Z = 0$$

which proves the claim. □

Definition 7. A two-sided module of an algebra $\mathcal{B}$ is the real vector space W which is endowed with the two-sided linear mappings

$$\mathcal{B} \times W \ni (a, w) \to a \cdot w \in W \quad \text{and} \quad W \times \mathcal{B} \ni (w, a) \to w \cdot a \in W. \tag{1}$$

In what follows, we use the following notations:

$$\begin{aligned} (a, b, w) &= (a \cdot b) \cdot w - a \cdot (b \cdot w), \\ (a, w, b) &= (a \cdot w) \cdot b - a \cdot (w \cdot b), \\ (w, a, b) &= (w \cdot a) \cdot b - w \cdot (a \cdot b). \end{aligned} \tag{2}$$

Definition 8. A two-sided module W is Kozsul-Vinberg module of a Koszul-Vinberg algebra $\mathcal{B}$ if it satisfies the following identities :

$$\begin{aligned}(a,b,w) &= (b,a,w),\\ (a,w,b) &= (w,a,b).\end{aligned} \tag{3}$$

Definition 9. Given a two-sided module KV module W on $\mathcal{A}$,the subspace of Jacobi elements of W , namely $J(W)$ is defined as follows:

$$J(W) = \{w \in W : (a,b,w) = 0 \forall a,b \in \mathcal{A}\}$$

Remark 1. The vector space $C^\infty(M)$ is a left KV module of $\mathcal{A}$ whose left action is defined by:

$$X \cdot f = \nabla_X f.$$

3.2. The Scalar KV valued cohomology

The scalar KV complex is the $\mathbb{Z}$-graded vector space $C^\infty(\mathcal{A},\mathbb{R}) = \bigoplus_q C^q(\mathcal{A}, C^\infty(M))$ by the following homogeneous subspaces :

$$C^q(\mathcal{A},\mathbb{R}) = 0 \text{ for } q < 0$$

$$C^0(\mathcal{A},\mathbb{R}) = \{f \in C^\infty(M) : \nabla^2 f = 0\}$$

$$C^q(\mathcal{A},\mathbb{R}) = Hom_{\mathbb{R}}(\mathcal{A}^{\otimes q}, C^\infty(M))$$

.

The coboundary operator is defined as it follows :

$$\delta^\nabla_{KV} f = -f$$

For $p > 0$,let $F \in C^p(\mathcal{A}, C^\infty(M))$:

$$\delta^\nabla_{KV} F(\xi) = \sum_{i \leqslant q} \nabla_{X_i} F(\partial_i \xi) - F(\nabla_{X_i} \partial_i \xi).$$

The cohomology space of this cochain is denoted by

$$H_{KV}(\mathcal{A}, C^\infty(M)) = \oplus_q H^q_{KV}(\mathcal{A}, C^\infty(M)).$$

Definition 10. A Hessian metric in a locally flat manifold (M,∇) is a non degenerate symmetric 2-cocyclic $g \in Z^2_{KV}(\mathcal{A}, C^\infty(M))$. It's cohomology class $[g] \in H^2_{KV}(\mathcal{A}, C^\infty(M))$ is called a non degenerate Hessian class.

Theorem 2. *Let g be a metric tensor in a locally flat manifold (M,∇).The following statements are equivalents.*
(1) (M,g,∇) is Hessian manifold.
(2) $\delta^\nabla g = 0$.
(3) Every point has a neighborhood $\mathcal{U}$ supporting a local smooth function f such that

$$g = \nabla^2 f$$

.

This theorem is a cohomological of proposition 2.1 as in [1]

3.3. The notion of Hyperbolicity of loclaly flat manifold

Given a locally flat manifold (M, ∇), we fix $x_0 \in M$ and we consider the space of differentiable paths

$$\{(0, [0,1]) \longrightarrow (x_0, M)\}$$

and we consider it's quotient modulo the fixed ends homotopy. This quotient set is denoted by $\tilde{M}$.

The map

$$\tilde{M} \ni [c] \rightarrow c(1) \in M$$

is a universal covering of M.

The universal covering of (M, ∇) is denoted by $(\tilde{M}, \tilde{\nabla})$. We consider a path c and $\theta \in [0, 1]$. Let τ_θ the parallel transport:

$$\tau_\theta : T_{x_0}M \rightarrow T_{c(\theta)}M$$

The developing map is defined as it follows:

$$D([c]) = \int_0^1 \tau_\theta^{-1} \frac{dc}{dt}(\theta) d\theta.$$

Definition 11. A locally flat manifold (M, ∇) is called hyperbolic if D is a diffeomorphism of $\tilde{M}$ onto a convex domain not containing any straight line (Koszul [6]).

Remark 2. This definition is equivalent to the fact that M is diffeomorphic to the orbit space

$$M = \frac{C}{D}$$

where C is an open convex domain not containing any straight line in $\mathbb{R}^n$ and D is a discrete subgroup of the Lie group $Aff(n)$ (see J. L. Koszul [6,7]).

Theorem 3. *(see [6]) For a locally flat manifold (M, ∇) being hyperbolic, it is necessary that it exists a de Rham closed differential 1-form ω on M whose covariant derivate $\nabla\omega$ is positive definite. If M is compact then this condition is also sufficient.*

Michel Boyom has given a cohomological analogue of the theorem above and his result states as follows:

Theorem 4. *[8] For a compact Hessian manifold (M, g, ∇), the following assertions are equivalent:*

1. $[g] = [0] \in H^2_{KV}(\mathcal{X}(M), C^\infty(M))$,
2. *(M,∇) is hyperbolic.*

4. The Hessian equation on a statistical manifold

4.1. Differential Operator

Let (V, p, M) a vector bundle and let k be any non negative integer. The vector bundle of k-jets of sections of vector bundle of V is denoted $J^k(V)$. At a point $x \in M$ the fiber $J^k_x(V)$ is quotient vector space:

$$J^k_x(V) = \frac{\Gamma(V)}{I^{k+1}_x(M)\Gamma(V)}$$

where $I_x(M) \subset C^\infty(M)$ is the ideal of differentiable functions that vanish at $x \in M$.

Let E, F two vectors bundles over M and $\Gamma(E)$ and $\Gamma(F)$ the spaces of their sections. For all k an nonnegative integer we denote j^k the map :

$$s \longrightarrow j^k(s)$$

from $\Gamma(E)$ to $\Gamma(J^k(E))$

Definition 12. The map

$$D : \Gamma(E) \longrightarrow \Gamma(F)$$

is k^{th} order differential operator if there exist an homomorphism of vector bundle

$$\mathscr{D} : J^k(E) \longrightarrow \Gamma(F)$$

such that

$$D(s) = \mathscr{D}(j^k s)$$

$\forall s \in \Gamma(E)$

$\mathscr{D}$ is called the principal symbol of D.

Let consider the short exact sequence

$$0 \to \tilde{J}^k(E) \xrightarrow{i} J^k(E) \xrightarrow{\pi^k} J^{k-1}(E) \to 0$$

Definition 13. The map

$$\sigma(D) = \mathcal{D} \circ i$$

from $\tilde{J}^k(E)$ to $\Gamma(F)$ is called the geometrical symbol of differential operator D

Let V and W be an finite vector spaces ,let ρ be a subspace of the vector space $Hom(V, W)$.

Definition 14. The first prolongation of ρ is subspace ρ^1 of $Hom(V, \rho)$ which is $\rho^1 = \rho \otimes V^\star \cap W \otimes S^2(V^\star)$.By recurrence we define on $p \in \mathbb{N}$,we defined the pth prolongation:

$$\rho^0 = \rho$$

$$\rho^p = V^\star \otimes \rho^{p-1} \cap S^{p+1}V^\star \otimes W$$

Definition 15. We consider the Koszul-Spencer complex of ρ:

$$\to Hom(\wedge^p V, \rho^q) \to Hom(\wedge^{p+1} V, \rho^{q-1}) \to$$

The differential operator ∂ is defined by

$$\partial f(v_1 \wedge .... \wedge v_{p+1} \otimes t_1.....t_q) = \sum_j (-1)^j f(v_1 \wedge ..... \wedge v_{p+1} \otimes v_j.t_1.....t_q)$$

Definition 16. ρ is called involutive if $H^{p,q}(\rho) = 0 \ \forall q, p$

4.2. Hessian equation of ∇

Given a gauge structure (M, ∇),we define the following second order differential operator

$$X \in \mathcal{X}(M) \to \nabla^2 X \in T^2_1(M)$$

Here $T^2_1(M)$ is the vector space of (2,1)-tensors in M.
We recall the definition of $\nabla^2 X$,

$$(\nabla^2 X)(Y, Z) = \nabla_Y(\nabla_Z X) - \nabla_{\nabla_Y Z} X$$

Definition 17. The Hessian equation of (M, ∇) is the following second differential equation

$$\nabla^2 X = 0$$

Let $\mathcal{J}_\nabla(M)$ the sheaf of solutions of Hessian equation $\nabla^2 X = 0$ and $J_\nabla(M)$ be the vector space of global section of the sheaf $\mathcal{J}_\nabla(M)$,it is well in [8] that $\mathcal{J}_\nabla$ is Associative Algebra sheaf(AAS) .
Let one evaluate the principal principal symbol of $X \to \nabla^2 X$. the principal symbol of the operator ∇^2 is defined by

$$(\nabla^2(X))(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \sum_k \Omega_{ij}^k \frac{\partial}{\partial x_k}$$

where

$$\Omega_{ij}^l = \frac{\partial^2 X^l}{\partial x_i \partial x_j} + \sum_k \left[\Gamma_{ik}^l \frac{\partial X^k}{\partial x_j} + \Gamma_{jk}^l \frac{\partial X^k}{\partial x_i} - \Gamma_{ij}^k \frac{\partial X^l}{\partial x_k} \right] + \sum_k \left[\frac{\partial \Gamma_{jk}^l}{\partial x_i} + \sum_m \left(\Gamma_{jk}^m \Gamma_{im}^k - \Gamma_{ij}^m \Gamma_{mk}^l \right) \right]$$

The geometrical symbol of ∇^2 denoted by $\sigma(\nabla^2)$ is

$$[\frac{\partial^2 X^k}{\partial x_i \partial x_j}]_{i,j} = [a_{i,j}^k]$$

$\sigma(\nabla^2)$ is equal to $S^2\mathbb{R}^{\star n} \otimes \mathbb{R}^n$, the first prolongation of $\sigma(\nabla^2)$ denoted $\sigma(\nabla^2)^1 = S^3\mathbb{R}^{\star n} \otimes \mathbb{R}^n$.It is easy to see that $\sigma(\nabla^2)$ admit a quasi regular basis then it is involutive, this is equivalent to $H^{p,q}(\sigma(\nabla^2)) = 0, \forall p, \forall q > 1$(see Jean Pierre Serre in appendix of [9]). We conclude that Hessian equation is involutive and finite type in sense of Elie Cartan [9],so we conclude that (J_∇, ∇) finite associative algebra and $(J_\nabla, [,]_\nabla)$ is finite Lie algebra.

Proposition 2. *Let (M, g) be a Riemmian manifold and let Ric be its Ricci curvature.We have*

$$J_\nabla \subset KerRic \tag{4}$$

Proof. Let $\xi \in J_\nabla$, by using the Bianchi identity we have:

$$\begin{aligned}
R^\nabla(\xi, X)Y &= R^\nabla(\xi, Y)X \qquad \forall X, Y \in \chi(M) \\
g(R^\nabla(\xi, X)Y, Z) &= g(R^\nabla(\xi, Y)X, Z) \\
&= -g(X, R^\nabla(\xi, Y)Z) \\
&= g(X, R^\nabla(\xi, Z)Y) \\
&= -g(R^\nabla(\xi, Z)X, Y) \\
&= g(R^\nabla(\xi, X)Z, Y) \\
&= -g(R^\nabla(\xi, X)Y, Z)
\end{aligned}$$

Then

$$J_\nabla \subset KerRic^\nabla$$

□

Corollary 1. *If ∇ is the Levi-Civita connection of an Einstein Riemanian manifold ,then*

$$J_\nabla = \{0\}$$

Definition 18. A gauge structure (M, ∇) is called special if

$$J_\nabla \neq \{0\}$$

Example 7: Every locally flat manifold (M, ∇) is special gauge structure.

Example 9: For any positive integer m ,$B^m = \{X \in \mathbb{R}^m / \|X\| < 1\}$ equipped with the Euclidean Levi-Civita connection is special gauge structure.
In the rest of this work we assume that our gauge structure are special.

Let $(M, g, \nabla, \nabla^\star)$ be a statistical manifold,consider the two Hessian operators ∇^2 and $\nabla^{\star 2}$ and we denote by $(J_\nabla(M), [,]_\nabla)$ and $(J_{\nabla^\star}(M), [,]_{\nabla^\star})$ the finite Lie algebra of global sections of sheaf of solutions of $\nabla^2 X = 0$ and $\nabla^{\star 2} X = 0$.
Let G_∇(resp $G_{\nabla^\star}$) a simply connected Lie group whose Lie algebra is denoted by J_∇(resp $J_{\nabla^\star}$) according to the third theorem of Lie.

Proposition 3. *Let $(M, g, \nabla, \nabla^\star)$ a statistical manifold and let $\{J_\nabla, J_{\nabla^\star}\}$ a pair of Lie algebras and $\{G_\nabla, G_{\nabla^\star}\}$ a pair of is Lie groups .We have the following properties:*

$$1)\{(J_\nabla, \nabla); (J_{\nabla^\star}, \nabla^\star)\}\textit{is a pair of KV-algebras},$$

$$2)\{(G_\nabla, \nabla); (G_{\nabla^\star}, \nabla^\star)\}\textit{pair of simply connected bi-invariant locally flat Lie groups}.$$

Definition 19. An infinitesimal dynamical of G_∇ is a manifold M is a lie algebra homomorphism from his Lie algbra J_∇ to the Lie algebra of vectors fields of M.

Definition 20. A locally effective action of lie group G_∇ on a differentiable manifold M is a group homomorphism from G_∇ to the diffeomorphism group of M such that his kernel is discrete.

Proposition 4. *J_∇ is integrable if is induced by a locally effective action of the Lie group G_∇.*
For more details see Richard S.Palais [10]

Remark 3.
1. If Any element of J_∇ is complete then J_∇ is integrable (If M is compact then J_∇ is integrable).
2. If (M, ∇) a geodesically complete locally flat manifold then J_∇ is integrable .

Definition 21. A locally flat foliation in a gauge structure (M, ∇) is a ∇-auto-parallel foliation $\mathcal{F}$ whose leaves are locally flat submanifolds on (M, ∇).

Definition 22. A Hessian foliation $\mathcal{F}$ in (M, g, ∇) is a locally flat foliation in gauge structure (M, ∇) such that the restriction of the Riemaniann metric on $\mathcal{F}$ is Hessiann metric.

Let $\mathcal{F}_\nabla$(resp $\mathcal{F}_{\nabla^\star}$) denoted the foliation defined by the G_∇-orbits(resp $G_{\nabla^\star}$).

Theorem 5. *Let $(M, g, \nabla, \nabla^\star)$ a compact statistical manifold. We have*
(1)The foliation $\mathcal{F}_\nabla$ is Hessian foliation in (M, g, ∇),
(2)The foliation $\mathcal{F}_{\nabla^\star}$ is Hessian foliation in $(M, g, \nabla^\star)$.

Proof. (1)By assumption M is compact then J_∇ is integrable,we conclude that J_∇ is infinitesimal action of G_∇ on M.The G_∇-orbits are ∇-auto-parallel and satisfy $R^\nabla(X, Y)J_\nabla = 0$ then $\mathcal{F}_\nabla$ is locally flat foliation in (M, ∇).The manifold (M, g, ∇) is statistical manifold and the G_∇-orbits are ∇-auto-parallel then the leaves of $\mathcal{F}_\nabla$ are Hessian submanifolds of (M, g, ∇).
(2) By using the same arguments for $G_{\nabla^\star}(M)$, the theorem is proved. □

5. Hyperbolicity Of Compacts Orbits

5.1. Basics Notions

Definition 23. Let G be a Lie group . A pair (f, q) of a homomorphism f from G to $GL(V)$ and a map q from G to V is said to be affine representation of G in V if it satisfies the following identity:

$$q(s_1 s_2) = f(s_1)q(s_2) + q(s_1)$$

124 $\forall s_1, s_2 \in G$

Definition 24. A affine representation of Lie algebra from $\mathcal{G}$ to V is homomorphism from the Lie algebra $\mathcal{G}$ to $aff(V)$.The affine representation of $\mathcal{G}$ to V is (f, q) such that f is linear map from $\mathcal{G}$ to $End(V)$ and q is linear map from $\mathcal{G}$ to V satisfy the following identities:

$$q([X, Y]) = f(X)q(Y) - f(Y)q(X), \forall X, Y \in \mathcal{G}$$

125 *5.2. The Canonical affine reprentation of affine Lie groups and is Lie algebra*

Let $G_\nabla = (G, \nabla)$ be a connected locally flat lie group whose (Left invariant) KV algebra is denote by J_∇ . In fact , the KV algebra of G_∇ is the real algebra

$$J_\nabla = (J, \nabla)$$

where J is Lie algebra of the lie group G and 1_{J_∇} is the identity endomorphism of the vector space J_∇ The affine representation of Lie algebra J_∇ in the vector space J_∇ by

$$J_\nabla \ni X \longrightarrow (\nabla, 1_{J_\nabla})(X) = (\nabla_X, X) \in aff(J_\nabla)$$

The affine map (∇_X, X) is defined by

$$(\nabla_X, X)(Y) = \nabla_X Y + X, \forall Y \in J_\nabla$$

Remark 4. $(\nabla, 1_{J_\nabla})$ is the differential at the unit element of a unique affine representation (f_∇, q_∇) of G_∇ to J_∇ ,namely

$$(f_\nabla, q_\nabla) : G_\nabla \longrightarrow Aff(J_\nabla)$$
$$\gamma \longrightarrow (f_\nabla(\gamma), q_\nabla(\gamma))$$

Where

$$(f_\nabla(\gamma), q_\nabla(\gamma))$$

is defined by :

$$(f_\nabla(\gamma), q_\nabla(\gamma))(X) = f_\nabla(\gamma)X + q_\nabla(\gamma)$$

and satisfy these identities :

$$\frac{d}{dt}(f(exptX), q(exptX))|_{t=0} = (\nabla_X, X)$$

and

$$q_\nabla(\gamma_1.\gamma_2) = f_\nabla(\gamma_1)q_\nabla(\gamma_2) + q_\nabla(\gamma_1)\forall\gamma_1, \gamma_2 \in G_\nabla$$

Remark 5. Let

$$f_\nabla : G_\nabla \longrightarrow GL(J_\nabla)$$
$$q_\nabla : G_\nabla \longrightarrow J_\nabla$$

Remind the cohomology of the Lie group G_∇ value in his Lie algebra J_∇ define by :

$$... \to C^q(G_\nabla, J_\nabla) \to C^{q+1}(G_\nabla, J_\nabla) \to C^{q+2}(G_\nabla, J_\nabla) \to$$

with differential operator d defined by :

$$d\theta(\gamma_1, ..., \gamma_{q+1}) = f_\nabla(\gamma_1).\theta(\gamma_2, ..., \gamma_{q+1}) + \sum_{i \leqslant q}(-1)^i\theta(.., \gamma_i\gamma_{i+1}, ...) + (-1)^q\theta(\gamma_1, ..., \gamma_q)$$

See Koszul[11] ,Cartan and Eleinberg [12] and Chevalley and Eleinberg [13],M.Boyom[14] for more details on Lie algebra cohomology.
It's easy to see that the condition :

$$q_{\nabla}(\gamma_1.\gamma_2) = f_{\nabla}(\gamma_1)q_{\nabla}(\gamma_2) + q_{\nabla}(\gamma_1)$$

$$\forall \gamma_1, \gamma_2 \in G_{\nabla}$$

is equivalent to the fact that $q_{\nabla} \in Z^1(G_{\nabla}, J_{\nabla})$, we denote $[q_{\nabla}] \in H^1(G_{\nabla}, J_{\nabla})$ is cohomology class

5.3. Vanishing Theorems and Hyperbocility

Let $(M, g, \nabla, \nabla^{\star})$ be a compact statistical manifold ,We go to deal conditions under which the Hessian foliations $\mathcal{F}_{\nabla}$ and $\mathcal{F}_{\nabla^{\star}}$ are hyperbolics.

Theorem 6. *Let $(M, g, \nabla, \nabla^{\star})$ a compact statistical manifold , the following assertions are equivalents on the leaves of $\mathcal{F}_{\nabla}$(resp on those of $\mathcal{F}_{\nabla^{\star}}$):*

$$\begin{aligned} &1)[g]_{\nabla} = 0 \in H^2_{KV}(J_{\nabla}, \mathbb{R}) \\ &2)[q_{\nabla^{\star}}] = 0 \in H^1(G_{\nabla^{\star}}, J_{\nabla^{\star}}) \\ &3)(f_{\nabla^{\star}}, q_{\nabla^{\star}}) : \textit{admit a fixed point} \\ &4)(f_{\nabla^{\star}}, q_{\nabla^{\star}}) \simeq f_{\nabla^{\star}} \end{aligned}$$

Proof. $(2) \to (1)$: If $[q_{\nabla^{\star}}] = 0$ then $\exists \xi \in J_{\nabla^{\star}}$ such that the following identities hold

$$q_{\nabla^{\star}}(\gamma_t) = f_{\nabla^{\star}}(\gamma_t)\xi - \xi, \forall \gamma_t \in G_{\nabla^{\star}}$$

by derivate this terms we obtain the following identity:

$$\frac{d}{dt}(q_{\nabla^{\star}}(\gamma_t))|_{t=0} = \frac{d}{dt}(f_{\nabla^{\star}}(\gamma_t)\xi - \xi)|_{t=0}, \forall \gamma_t \in G_{\nabla^{\star}}$$

$$X = \nabla^{\star}_X \xi, \forall X = \frac{d(\gamma_t)}{dt}|_{t=0}$$

by using the conjugaison formula, we deduce the following identity:

$$g(X, Y) = X.g(\xi, Y) - g(\xi, \nabla_X Y)$$

$$g(X, Y) = (d^{\nabla}_{KV}(i_{\xi} g))(X, Y)$$

$$[g]_{\nabla} = 0$$

by the same arguments we prove that $[q_{\nabla^{\star}}] = 0$ then $[g]_{\nabla} = 0$
$(1) \to (2)$:
Assume that $[g]_{\nabla} = 0$ then there exist a one form Θ define by $\Theta(X) = g(\xi, X)$ such that

$$g(X, Y) = -X\Theta(Y) + \Theta(\nabla_X Y)$$

by using the conjugaison we obtain the following identity

$$g(\nabla^{\star}_X \xi, Y) = X.g(\xi, Y) - g(\xi, \nabla_X Y)$$

we deduce the following relation:

$$\nabla^{\star}_{X}(-\xi) = X$$

$\forall \gamma_t \in G_{\nabla^\star}$ such that $q_{\nabla^\star}(\gamma_t) = f_{\nabla^\star}(\gamma_t)\xi - \xi$, then we conclude that

$$[q_{\nabla^\star}] = 0 \in H^1(G_{\nabla^\star}, J_{\nabla^\star})$$

$$2) \iff 3) \iff 4)$$

his well known in[15][16]
<u>Hint</u> $(2) \rightarrow (3)$ Assume that (2) $q_{\nabla^\star} \in B^2(G_{\nabla^\star}, J_{\nabla^\star})$ then $\exists \xi \in J_{\nabla^\star}$ such that:

$$f_{\nabla^\star}(\gamma)(-\xi) + q_{\nabla^\star}(\gamma) = -\xi$$

then $(f_{\nabla^\star}, q_{\nabla^\star})$ admit an fixed point.
Now we assume $(3) \rightarrow (4)$

$$\begin{bmatrix} 1 & \xi \\ 0 & 1 \end{bmatrix} \begin{bmatrix} f_{\nabla^\star} & q_{\nabla^\star} \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & -\xi \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} f_{\nabla^\star} & 0 \\ 0 & 1 \end{bmatrix} \tag{5}$$

$(4) \rightarrow (2)$ Assume that $< f_{\nabla^\star}, q_{\nabla^\star} >$ is linearisable then $\exists \xi \in J_{\nabla^\star}$ such that

$$q_{\nabla^\star}(\gamma) = f_{\nabla^\star}(\gamma)\xi - \xi, \forall \gamma \in G_{\nabla^\star}$$

$$q_{\nabla^\star}(\gamma) = d(\xi)(\gamma)$$

we conclude that $[q_{\nabla^\star}] = [0]$. □

Let denoted by F the compact leaf of foliation $\mathcal{F}_{\nabla}$ and $F^\star$ those of $\mathcal{F}_{\nabla^\star}$.

Corollary 2. *Let $(M, g, \nabla, \nabla^\star)$ be a compact statistical manifold.Under the hypotheses of Theorem 6, we have :*
(1) The compacts leaves (F, ∇) are hyperbolics manifolds,
(2) The compacts leaves $(F^\star, \nabla^\star)$ are hyperbolics manifolds.

<u>Comments</u>
In[8] Boyom proves that the compacts orbits of $G_{\nabla^{LC}}$ in case of trivial statistical manifold (M, g, ∇^{LC}) are flat cylinder over Riemaniann torus.$(Orb(G_{\nabla^{LC}})(p), g) \simeq (\frac{T^{b_1(Orb(G_{\nabla^{LC}})}}{K} \times \mathbb{R}^{r(\nabla^{LC})(p)-b_1((Orb(G_{\nabla^{LC}}))}, g_0))$.Here $r(\nabla^{LC})(p)$ is the dimension of the orbits on p, g_0 is Euclidean (flat)metric of $\mathbb{R}^{r(\nabla^{LC})p}$,$b_1(Orb(G_{\nabla^{LC}}))$ is the first Betti number of $Orb(G_{\nabla^{LC}})$,K is finite subgroup of isometrie of $g_0$ie($Iso(g_0) \simeq O(r(\nabla^0)p) \ltimes \mathbb{R}^{r(\nabla^{LC})p}$)[17].
We proves the similary result with hyperbolic orbits of G_{∇} $(Orb(G_{\nabla}(p), \nabla) \simeq (\frac{U}{\Gamma}, \nabla^F)$.Here U s open convex domain not containing any straight line in $\mathbb{R}^{r(\nabla)p}$,Γ is discrete a subgroup of $Aff(\mathbb{R}^{r(\nabla)p})$ie($GL(\mathbb{R}^{r(\nabla)p} \ltimes \mathbb{R}^{r(\nabla)p})$,$\nabla^F$ the canonical Flat connection on $\mathbb{R}^{r(\nabla)p}$.

Now we discuss about somes conditions under which the universal covering of compacts hyperbolics orbits are convex cones. Let remind an important theorem of J.L.Koszul.

Theorem 7. *[18] If the universal covering of a compact hyperbolic locally flat manifold is affinely homogeneous, then his universal covering is isomorphic to a convex cone.*

Theorem 8. *Let $(M, g, \nabla, \nabla^{\star})$ a compact statistical manifold which satisfy one of the equivalents conditions of theorem 6,we have the following assertions.*
1)The compacts leaves of $\mathcal{F}_{\nabla}$ are isomorphic to the quotient of convex homogeneous cones by discrete subgroups of $Aff(\mathbb{R}^{r(\nabla)^{p}})$,
2)The compacts leaves of $\mathcal{F}_{\nabla^{\star}}$ are isomorphic to quotient of convex homogeneous cones by discrete subgroups of $Aff(\mathbb{R}^{r(\nabla^{\star})^{q}}))$.

Proof. By using the following formula on the G_{∇}-Orbit :

$$\mathcal{L}_{\xi}\nabla = i_{\xi}R^{\nabla} + \nabla^{2}\xi$$

we conclude that $G_{\nabla} = Aff(Orbit(G_{\nabla}), \nabla)$ .The Lie Group $Aff(Orbit(G_{\nabla}), \nabla)$ is isomorphic to the normalizer $N(\pi_1(Orbit(G_{\nabla}))$ of $\pi_1(Orbit(G_{\nabla}))$ in $Aff(\tilde{Orbit}(G_{\nabla}), \tilde{\nabla})$,then we conclude that G_{∇} acts affinely on $(\tilde{Orbit}(G_{\nabla}), \tilde{\nabla})$.The action of G_{∇} on Orbit(G_{∇}) is affinely transitive ,then the orbits of this action at $\tilde{x}$ denote by $G_{\nabla}(\tilde{x})$ is isomorphic the Orbit(G_{∇}), then dim($G_{\nabla}(\tilde{x})$)=dim($\tilde{Orbit}(G_{\nabla})$).We conclude that $G_{\nabla}(\tilde{x})$ is Open subset of $\tilde{Orbit}(G_{\nabla})$,but $\tilde{Orbit}(G_{\nabla})$ is connected then G_{∇} acts transitively on $\tilde{Orbit}(G_{\nabla})$.By using the Theorem 7 ,we conclude that the compact orbits $Orbt(G_{\nabla})$ are isomorphic to the Quotient of Convex cone by the discrete subgroup Γ of $Aff(\mathbb{R}^{dim(Orbit(G_{\nabla})})$.
We can do the same arguments for $G_{\nabla^{\star}}$. □

Remark 6. Under Barbaresco formalisms the universal coverings of the compacts leaves of our hyperbolics foliations are statistical models.

Corollary 3. *If $(M, g, \nabla, \nabla^{\star})$ is a statistical model (Θ, P) then it's foliated by leaves which are quotient of exponential models.*

Example:

We shall give a example of compact hessian manifold which orbits are quotient of statistical models by discrete subgroup.Let $\mathbb{R}^2$ be a 2-dimensional real affine space with the natural flat affine connection D and let $\{x, y\}$ be an affine coordinate system of $\mathbb{R}^2$.
Let Ω be a domain define by $x > 0$ and $y > 0$.Consider a Riemannian metric on Ω given by $g = \frac{1}{x^2}dx^2 + \frac{1}{y^2}dy^2$.
Then (D,g) is Hessian structure on Ω.
Let Σ and χ be linear transformations on Ω defined by :

$$\Sigma : (x, y) \to (2x, y)$$

$$\chi : (x, y) \to (2x, 3y)$$

Then <Σ,χ> leave the Hessian structure (D, g) invariant.
We denote Γ the group generate by $\{\Sigma, \chi\}$, we can also write Γ as $< \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} ; \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} >$.
Γ acts properly discontinuously on Ω and $\frac{\Omega}{\Gamma}$ is is compact Hessian manifold which is diffeomorphic to a Torus.
Let us denote by π the projection from Ω to $\frac{\Omega}{\Gamma}$ and by (D, g) the Hessian structure on $\frac{\Omega}{\Gamma}$.Since the space of all Γ-invariant D parallel 1-forms on $\Omega$is spanned by dx and dy.The space of all D-parallel 1-forms on $\frac{\Omega}{\Gamma}$ is spanned by ω and Φ where $dx = \pi^{\star}\omega$ and $dy = \pi^{\star}\Phi$.Let X and $\tilde{X}$ be a vector field on $\frac{\Omega}{\Gamma}$ defined by $\omega(Y) = g(X, Y)$ and $\Phi(Y) = g(\tilde{X}, Y)$for each vector field Y on $\frac{\Omega}{\Gamma}$.Then $X = \pi_{\star}(\frac{\partial}{\partial x})$ and $\tilde{X} = \pi_{\star}(\frac{\partial}{\partial y})$ and the vector space J_{∇} of all Hessian vector fields on $\frac{\Omega}{\Gamma}$ is spanned by <X,$\tilde{X}$>.Since T^2 is compact the J_{∇} is integrable.Let $Exp(tX)$ and $Exp(tY)$ is a 1-parameter group of transformations generated by X and by Y.Let $G_{\nabla} =< Exp(tX), Exp(tY) >$. Consider $a > 0$ and $b > 0$, the compact homogeneous the orbit $G_{\nabla}\pi(a, b) = \{\gamma.\pi(a, b), \gamma \in G_{\nabla}\}$ is a circle. Then we conclude that (S^1, D, g_{S^1}) is Hessian manifold and satisfy $D\frac{\partial}{\partial\theta} = 0$ and $g_{S^1} = \frac{1}{\theta^2}d\theta^{\otimes 2}$.By using conjugation we have $D^{\star}_{\frac{\partial}{\partial\theta}}\frac{\partial}{\partial\theta} = -\frac{2}{\theta}\frac{\partial}{\partial\theta}$.Let

$H^\star = h(\theta)\frac{\partial}{\partial\theta}$ vectors fields on S^1.$H^\star$ is homothety vector fields of $D^\star$ if h is solution of differential equation $y' - \frac{2}{\theta^2}y = 1$.We deduce that

$$h(\theta) = Ae^{\frac{-2}{\theta}} + 2e^{\frac{-2}{\theta}} \int_{\frac{-2}{\theta}}^{+\infty} \frac{e^{-s}}{s} \mathrm{d}s + \theta$$

where A is constant.
As $D^\star H^\star = Id$ then (S^1, D) is hyperbolic.Now let $p : \mathbb{R} \to S^1$ the covering map, and consider the map

$$(\mathbb{R}, \tilde{D}) \to (S^1, D)$$

.The compacts orbits is $G_\nabla \pi(a,b)$ is quotient $\frac{\mathbb{R}^{>0}}{\Gamma}$, here Γ is discrete subgroup of $Gl(\mathbb{R}) \ltimes \mathbb{R}$

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