Cliques and a New Measure of Clustering
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To cite this version:
Steve Lawford, Yll Mehmeti. Cliques and a New Measure of Clustering. CCS 2020, Dec 2020, Virtual event, France. ACM. hal-03142525

HAL Id: hal-03142525
https://hal-enac.archives-ouvertes.fr/hal-03142525
Submitted on 16 Feb 2021

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1. INTRODUCTION

One widely used measure of clustering is the overall clustering coefficient, or “transitivity”, on three nodes:

$$C(3) = \frac{3 \times \text{number of triangles in the network } G}{\text{number of connected triples of nodes in } G}$$

which quantifies the relative frequency with which two neighbours of a node are themselves neighbours.

Many real-world networks display higher levels of clustering than if those networks were random [1, 2].

Clustering related to cooperative social behaviour and beneficial information and reputation transfer [3].

Significant topological structures, on more than three nodes, can be found in real-world networks, and may perform much better than expected [4].

A generalized clustering coefficient could provide new insight in such higher-order network structure.

2. OBJECTIVES

a) Propose a higher-order generalization of $C(3)$, for any number of nodes, that nests standard clustering.

b) Develop and test a fast, practical, implementation based on analytic subgraph enumeration formulae.

3. HIGHER-ORDER CLUSTERING

We define the generalized clustering coefficient as:

$$C(b) = \frac{a(b) \cdot \text{number of b-cliques } K_b \text{ in } G}{\text{number of b-spanning trees in } G}$$

where $Cagley$’s formula $a(b) = b^{b-2}$ gives the number of spanning trees in $K_b$, ensuring that $0 \leq C(b) \leq 1$.

We use analytic subgraph enumeration formulae to count cliques and spanning trees [5, 6]:

$$C(4) = \frac{16 |K_4|}{|M_{12}^{(4)}| + |M_{13}^{(4)}|}$$

$$C(5) = \frac{125 |K_5|}{|M_{12}^{(5)}| + |M_{13}^{(5)}| + |M_{14}^{(5)}|}$$

where $|M_{ij}^{(k)}|$ is the count of subgraphs of “type” $a$ on $b$ nodes. For example, the 3-arrow subgraph count is:

$$|M_{13}^{(4)}| = \sum_{(i,j) \in E} \left( \frac{K_3 - 1}{2} (k_3 - 1) - 2 |M_{ij}^{(4)}| \right)$$

where edge $(i,j) \in E$ is summed in both directions, $k_3$ is the node degree, and $|M_{ij}^{(4)}|$ is the tadpole count.

An alternative measure was developed in 2018 by Yin-Benson-Leskovec (YBL), using clique expansion [7].

$$C_{b-1} = \frac{(2^b - b) |K_b|}{|L( b - 1, 1)|} \quad b \geq 4,$$

where $L(\cdot, \cdot)$ is the lollipop graph formed by joining a $(b-1)$-clique by a bridge to a single node.

Critical difference between $C(b)$ and $C_{b-1}$ is in their definitions of the “relative frequency” of cliques.

4. THEORETICAL RESULTS ON RANDOM GRAPHS

5. EMPIRICAL RESULTS ON REAL-WORLD NETWORKS

Figure 1: Theoretical difference in expectation for the Erdős-Rényi random graph $G(n, p)$ is $E[G(C_b) - E[G(C_0)] = p(b^2 - 1) - (b - 1)^2 b^{b-3}$, with edge-formation probability $p$.

Figure 2: Simulated expected clustering $E[G(C_b)]$ from 250 replications of a small-world graph with $n = 50$ nodes, each of which has degree 14, and edge-rewiring probability $p$.

Figure 3: Descriptive statistics for 2013Q4. The average path lengths (apl) for real-world networks are close to those from Erdős-Rényi random graphs ($\frac{g^2}{q}$).

6. DISCUSSION AND FUTURE DIRECTIONS

a) Our work complements YBL: (theory) with $C(b)$, we develop the other natural generalization of $C(3)$ to more nodes, (computational) we derive analytic higher-order clustering formulae, while YBL use numerical methods, (empirical) we apply $C(b)$ to airline networks, a classical example that is not covered by YBL.

b) It is hard to derive analytic count formulae for subgraphs as $b$ increases e.g. $C(8)$ has 23 denominator terms. There may be a role for computer-assisted (or automated) theorem proving in working towards this goal.

c) Airline carriers are increasingly developing small groups of highly-connected airports. The concept of a “hub” (or central) node in real-world networks can be extended to “multi-node hubs” (or central groups of nodes).

REFERENCES


CONTACT INFORMATION

I will be happy to discuss problems, papers and projects in all areas of complex systems after CCS2020.

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My complex systems papers are at: http://tinyurl.com/arxiv-steve

Scan the QR code (top-right) for the clustering paper!