CANONICAL FOLIATIONS OF STATISTICAL MANIFOLDS WITH STATISTICAL MODELS

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ABSTRACT. The sheaf of solutions $\mathcal{J}_\nabla$ of the Hessian equation on a gauge structure (M, ∇) is a key ingredient for understanding important properties from the cohomological point of view. In this work, a canonical representation of the group associated by Lie third's theorem to the Lie algebra formed by the sections of $\mathcal{J}_\nabla$ is introduced. On the foliation it defines, a characterization of compact hyperbolic leaves is then obtained. We conclude that these leaves are equipped with a statistical model structure.

1. INTRODUCTION

Hyperbolicity is quite an important notion in geometry and can be tackled using different approaches. In its original work [11], Koszul introduces the development map Q to define hyperbolic manifold as those for which the image by Q of their universal coverings is an open convex domain without straight line. In [13], a cohomological characterization is given, stating that a necessary condition for a gauge structure (M, ∇) to be hyperbolic is the existence of a closed 1-form admitting a positive definite covariant derivative. Finally, if (M, g, ∇) is a gauge structure on an Hessian manifold, then (M, ∇) is hyperbolic iff $[g] = 0$ in Koszul-Vinberg cohomology [3].In [17], Shima observes that Fisher information of all classical used statistical models are Hessian metrics. As Amari did, he also proved that any locally flat statistical manifold is a Hessian manifold. These structures arised from works of many mathematicians (e.g. J.-L. Koszul, Y. Matsushima, A. Nijenhuis, E. B. Vinberg) in their attempt to solve the Gerstenhaber conjecture in the category of locally flat manifold.The geometry of locally flat hyperbolic manifold is also named Koszul geometry (see [8]),the conjecture of Gerstenhaber say that every restricted theory of deformation generates its proper theory of cohomology.This conjecture has been solved in 2006 (see [14]) by Michel Boyom. In [3, 15] he also realize the homological version of the Hessian geometry.The aim of this paper is to study hyperbolicity of the leaves of canonical foliations in statistical manifolds using representations in the affine group of a finite dimensional vector space and equipped these leaves with statistical model.The paper is organized as follows:in section (2), we summarize the main results,section 3 is devoted to basics notions which are used,in section (4), is devoted to useful notions in the category of locally flat manifolds and their KV-cohomology. In section (5) we introduce the notion of hyperbolicity in sense of Koszul, Vey.In section (6), the Hessian equation on a gauge structure is introduced along with some basic facts. In section (7), the canonical group representation is defined, from which a vanishing condition in cohomology is given in Theorem 7.2. Finally, the case of statistical manifolds is treated in section (8) and a characterization of hyperbolic compact leaves is given.In section (9) we prove that the universal covering of these leaves are convex cones.In the last section (10) we show that leaves are bearing a statistical model structure pertaining to the exponential family. In the final section (11), a explicit construction is given and conclusion drawn.

2. Main results

The principal theorems proved in this paper are summarized below.

Theorem 2.1. *The following statements are equivalent :*

(1) *The affine action*
$$G_\nabla \ltimes J_\nabla \ni (\gamma, X) \to f_\nabla(\gamma).X + q_\nabla(\gamma) \in J_\nabla$$
has a fixed point;

(2) *The cohomology class* $[q_\nabla]$ *vanishes;*

(3) *The affine representation*
$$G_\nabla \ni \gamma \to (f_\nabla(\gamma), q_\nabla(\gamma) \in Aff(J_\nabla)$$
is conjugated to the linear representation
$$G_\nabla \ni \gamma \to f_\nabla(\gamma) \in GL(J_\nabla).$$

Theorem 2.2. *Let* $(M, g, \nabla, \nabla^\star)$ *be a compact statistical manifold. Then the foliation* $\mathcal{F}_\nabla$ *(resp.* $\mathcal{F}_{\nabla^*}$*) is a Hessian foliation in* (M, g, ∇) *(resp.* (M, g, ∇^*)*).*

Theorem 2.3. *In a statistical manifold* $(M, g, \nabla, \nabla^\star)$ *the following assertions are equivalent:*

(1) $[g_\nabla] = 0 \in H^2_{KV}(\mathcal{F}_\nabla)$

(2) $[q_{\nabla^\star}] = 0 \in H^1(G_{\nabla^\star}, J_{\nabla^\star})$

(3) $(f_{\nabla^\star}, q_{\nabla^\star})$ *has a fixed point.*

(4) $(f_{\nabla^\star}, q_{\nabla^\star})$ *is affinely conjugated to its linear component* $f_{\nabla^\star}$*.*

Theorem 2.4. *Let* (M, g, ∇) *being a compact Hessian manifold. If the characteristic obstruction satisfies:*
$$\{[g_\nabla] \cup [q_{\nabla^\star}]\} = \{0\} \cup \{0\}.$$
Then (M, ∇) *is an hyperbolic manifold.*

Theorem 2.5. *Let* (M, ∇) *be a closed hyperbolic manifold.Then*
$$[q_{\nabla^\star}] = 0 \in H^1(G_{\nabla^\star}, J_{\nabla^\star})$$
and
$$[q_\nabla] = 0 \in H^1(G_\nabla, J_\nabla)$$

Theorem 2.6. *Let* (M, ∇) *be a closed hyperbolic manifold.Then* $(M, \nabla^\star)$ *is also hyperbolic manifold.*

Theorem 2.7. *Let* (N, g, ∇) *be a statistical manifold and let* $\mathcal{F}_\nabla$ *be its canonical Hessian foliation.*
If the characteristic obstruction vanishes:
$$\{[g_\nabla] \cup [q_{\nabla^\star}]\} = \{0\} \cup \{0\}.$$
Then up to a affine diffeomorphism every compact leaf of $\mathcal{F}_\nabla$ *is the quotient of a sharp open convex cone by a discrete group of* $GL(\mathbb{R}^n)$

Theorem 2.8. *Let (N, g, ∇) be a statistical manifold and let $\mathcal{F}_\nabla$ be its canonical Hessian foliation. If the characteristic obstruction vanishes:*

$$\{[g_\nabla] \cup [q_{\nabla^\star}]\} = \{0\} \cup \{0\}.$$

Then up to affine diffeomorphism, every compact leaf of $\mathcal{F}_\nabla$ is a parametric space of statistical models for a measurable set $(\Omega^\star, \mathcal{B}(\Omega^\star))$

$$< F_\nabla, Q >$$

,where Q is defined by

$$Q(\Theta, \xi) = \int_{H(\nabla)} \frac{e^{-<\gamma\Theta,\xi>}}{\int_{\Omega^\star} e^{-<\gamma\Theta,\xi>} d\xi} d\mu^{H(\nabla)}$$

with $H(\nabla) \subset GL(\mathbb{R}^n)$

3. Basic notions

Most of the material presented here can be found in greater detail in [3]. Only the required notions will be introduce in this section. In the sequel, $\chi(M)$ stands for the real Lie algebra of smooth vector fields on M. The convention of summation on repeated indices will be used thorough the document.

Definition 3.1. *A Koszul connection is a first order differential operator ∇ of $TM^{\star\otimes 2}$ to TM. It is usually denoted by*

$$(X, Y) \to \nabla_X Y$$

and has the following properties:

(1)
$$\nabla_{fX} Y = f \nabla_X Y$$

(2)
$$\nabla_X fY = df(X)Y + f\nabla_X Y \quad for f \in C^\infty(M), \forall X, Y \in \mathcal{X}(M)$$

The torsion tensor T^∇ and the curvature tensor R^∇ are defined by:

(1)
$$T^\nabla(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y]$$

(2)
$$R^\nabla(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

Definition 3.2. *A gauge structure is a pair (M, ∇) where ∇ is a Koszul connection in M.*

4. Locally flat structure on M and KV-cohomology

Definition 4.1. *An affinely flat structure in a m-dimensional manifold M is defined by a complete atlas:*

$$\mathcal{A} = \{(U_i, \Phi_i)\}$$

whose local chart changes $\Phi_i^{-1} \circ \Phi_j$ are restrictions of affine transformations to the m-dimensional affine space $\mathbb{R}^m$.

Definition 4.2. *A locally flat manifold is a pair (M, ∇) where ∇ is a linear Koszul connection whose curvature R^∇ and torsion T^∇ both vanish identically:*

(1)
$$\nabla_X Y - \nabla_Y X - [X,Y] = 0$$

(2)
$$\nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z = 0$$

4.1. **KV-cohomology of locally flat manifolds.** Let (M,∇) be a locally flat manifold.Let $\mathcal{A} = (\mathcal{X}(M),\nabla)$ be the Koszul-Vinberg algebra associated to (M,∇).For a locally flat manifold the Koszul-Vinberg complex is defined as follows:

Definition 4.3. *Let $C_q(\nabla), q \geq 0$ be the vector spaces:*

$$\begin{cases} C^0(\nabla) = \{f \in C^\infty(M),\ \nabla^2 f = 0\} \\ C^q(\nabla) = Hom_{\mathbb{R}}(\otimes^q \chi(M), C^\infty(M)) \end{cases}$$

and let $\delta\colon C^q(M) \to C^{q+1}(M)$ be the coboundary operator:

$$\begin{cases} \delta f = df\, \forall f \in C^0(\nabla) \\ \delta f(X_1 \otimes \ldots \otimes X_{q+1}) = \Sigma_1^q (-1)^i [d(f(.. \otimes \hat{X}_i \otimes .. \otimes X_{q+1}))(X_i) \\ \qquad - \Sigma_{j \neq i} f(.. \otimes \hat{X}_i \otimes .. \otimes \nabla_{X_i} X_j \otimes ..] \end{cases}$$

The complex $(C(\nabla),\delta)$ is the Koszul-Vinberg complex of (M,∇), with cohomology groups denoted by $H^q_{KV}(\nabla)$.

Definition 4.4. *An element g in $H^2_{KV}(\nabla)$ is called a Hessian (resp. Hessian non degenerate, Hessian positive definite) class if it contains a symmetric (resp symmetric non degenerate, symmetric positive definite) cocycle.*

4.2. **Hessian manifold.**

Proposition 4.1. *[18] Let g be a metric tensor in a locally flat manifold (M,∇). The following statements are equivalent;*

(1) *(M,g,∇) is Hessian manifold*
(2) *Every point has a neighborhood U supporting a local smooth function h such that*
$$g = \nabla^2 f$$

The following proposition gives a cohomological definition of a Hessian manifold .

Proposition 4.2 ([14]). *A locally flat manifold (M,g,∇) is Hessian manifold in the sense of [18] if δg vanishes identically .*

5. Hyperbolicity of locally flat manifold

The opposite of geodesic completeness is hyperbolicity in the sense of Vey [[20]].Hyperbolic affine manifolds are closely related to Hessian manifold.This notion of hyperbolicity has been studied by many authors Koszul[[13], [11], [12]],Vey[[20], [22]],Kobayashi[[9], [10]].The notion of hyperbolicity is among the fundamental notions of the geometry of Koszul.It may be addressed and studied form many perspectives. We highlight here the algebraic topology point of view, which is richer than the Riemannian geometry one. As an example, the theory of affine representations of Lie groups plays a key role in Lie group theory of Heat after Jean-Marie Souriau.

5.1. **Developing map of locally flat manifold.** Given a locally flat structure (M, ∇), we fix a point $x_0 \in M$ and we consider the pairs $\{0, [0, 1]\}$ and $\{x_0, \mathrm{M}\}$. We go to deal with the space of differentiable paths

$$c : \{0, [0, 1]\} \rightarrow \{x_0, \mathrm{M}\}\,.$$

The notation means that

$$c(0) = x_0.$$

The quotient modulo the fixed ends homotopy is denoted by $\tilde{\mathrm{M}}$.
The application

$$\tilde{\mathrm{M}} \ni [c] \rightarrow \pi([c]) = c(1) \in \mathrm{M}$$

is is a universal covering of M.The universal covering of (M, ∇) is denoted by $(M, \tilde{\nabla})$.
Consider a path c,$\sigma \in [0, 1]$,let τ_σ be the parallel transport

$$\tau_\sigma : \mathrm{T}_{x_0}\mathrm{M} \rightarrow \mathrm{T}_{c(\sigma)}\mathrm{M}.$$

The developing map D is defined as it follows:

$$D : \mathrm{M} \ni [c] \rightarrow D([c]) = \int_0^1 \tau_\sigma^{-1}(\frac{dc}{dt}(\sigma) d\sigma \in \mathrm{T}_{x_0}\mathrm{M}$$

Definition 5.1. *A locally flat manifold (M, ∇) is called hyperbolic if D is a diffeomorphism of $\tilde{M}$ onto a convex domain not containing any straight line (Koszul [13]).*

Remark 5.2. *This definition is equivalent to the fact that M is diffeomorphic to the orbit space*

$$M = \frac{C}{D}$$

where C is an open convex domain not containing any straight line in $\mathbb{R}^n$ and D is a discrete subgroup of the Lie group $Aff(n)$ (see J. L. Koszul [13, 20]).

Theorem 5.3. *(see [13]) For a locally flat manifold (M, ∇) being hyperbolic, it is necessary that it exists a de Rham closed differential 1-form ω on M whose covariant derivate $\nabla\omega$ is positive definite. If M is compact then this condition is also sufficient.*

Remark 5.4. *The existence of the nonvanishing Koszul 1-form ω proves that M is fibered over $\mathbb{S}^1$ [19].Topological consequences follow,the Euler characteristic $\mathcal{X}(M) = 0$ and the first Betti number $b_1(M) \geq 1$.*

In [4] Michel Boyom has given a cohomological analogue of the theorem above and his result states as follows:

Theorem 5.5. *Let (M, g, ∇) be a compact Hessian manifold then the following assertions are equivalent:*

- $[g] = 0 \in H^2_{KV}(\nabla)$.
- (M, ∇) *is hyperbolic*

6. Hessian equation of ∇

Definition 6.1. *The Hessian operator $\nabla^2 : \chi(M) \to T^1_2(M)$ is, for a fixed $Z \in \chi(M)$, the covariant derivative of the $T^1_1(M)$-tensor ∇Z. For any triple (X, Y, Z) of vector fields, its expression is given by:*

$$\nabla^2_{X,Y} Z = \nabla_X(\nabla_Y Z) - \nabla_{\nabla_X Y} Z.$$

Proposition 6.1. *The product $(X, Y) \in \chi(M) \mapsto \nabla_X Y$ has associator ∇^2*

Proof. This a direct consequence of def. 6.1, since:

$$\nabla^2_{X,Y} Z = X.(Y.Z) - (X.Y).Z$$

□

Definition 6.2. *The Hessian equation ∇ is defined by:*

$$HE(\nabla): \quad \nabla^2 X = 0$$

6.1. **Local expression of Hessian equation of ∇.** Let $(x_1, .., x_m)$ be a system of local coordinate functions of M and let $X \in \mathcal{X}(M)$. We set

$$\partial i = \frac{\partial}{\partial x_i},$$

$$X = \sum X^k \partial k$$

$$\nabla_{\partial i} \partial j = \Sigma_k \Gamma^k_{i.j}$$

The principal of symbol of the Hessian differential operator can be expressed as

$$(\nabla^2 X)(\partial i, \partial j) = \Sigma_k \Omega^k_{i,j}(X) \partial k \tag{6.1}$$

where

$$\begin{aligned} \Omega^l_{ij}(X) = & \frac{\partial^2 X^l}{\partial x_i \partial x_j} + \Gamma^l_{ik} \frac{\partial X^k}{\partial x_j} + \Gamma^l_{jk} \frac{\partial X^k}{\partial x_i} - \Gamma^k_{ij} \frac{\partial X^l}{\partial x_k} \\ & + \frac{\partial \Gamma^l_{jk}}{\partial x_i} + \Gamma^m_{jk} \Gamma^k_{im} - \Gamma^m_{ij} \Gamma^l_{mk} \end{aligned} \tag{6.2}$$

The geometric symbol σ_∇ of ∇^2 is

$$\sigma_\nabla(X)(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) = \Sigma_l \frac{\partial^2 X^l}{\partial x_i \partial x_j} \frac{\partial}{\partial x_l}$$

he Hessian differential operator ∇^2 is involutive (The Koszul-Spencer complex is acyclic).

Definition 6.3 ([4]). *Let (M, ∇) be a gauge structure. The sheaf of solutions of its Hessian equation, denoted by $\mathcal{J}_\nabla(M)$, is the sheaf of associative algebras:*

$$U \mapsto \{X \in \chi(U),\ \nabla^2 X = 0\}$$

with product defined in prop. 6.1.

The space of sections of $\mathcal{J}_\nabla(M)$ will be denoted in the sequel by J_∇.

Proposition 6.2. *The pair* (J_∇, ∇) *is an associative algebra with commutator Lie algebra* $(J_\nabla, [-,-]_\nabla)$ *where the bracket* $[X,Y]_\nabla$ *is:*

$$[X,Y]_\nabla = \nabla_X Y - \nabla_Y X.$$

When ∇ has vanishing torsion, (M, ∇) is said to be a symmetric gauge structure and the Lie algebra $(J_\nabla, [-,-]_\nabla)$ is obviously a Lie subalgebra of the Lie algebra of vector fields

$$(\mathcal{X}(M), [-,-]);$$

Proposition 6.3 ([4])**.** *If* (M, ∇) *is symmetric gauge structure, then the Lie subalgebra* $J_\nabla \subset \mathcal{X}(M)$ *is finite-dimensional over the field of real numbers.*

The next propositions shows that J_∇ may be trivial.

Proposition 6.4. *Let (M,g) be a Riemmian manifold and let Ric be its Ricci curvature tensor.It comes:*

$$J_{\nabla^{LC}} \subset ker(Ric)$$

Proof. Let (X,Y) be a couple of vector fields on M and let $\xi \in J_{\nabla^{LC}}$.We have:

$$R^{\nabla^{LC}}(X,Y)\xi = (\nabla^2\xi)(X,Y) - (\nabla^2\xi)(Y,X)$$

Taking the trace we deduce that:

$$Ric(X,\xi) = 0$$

Then

$$J_{\nabla^{LC}} \subset ker(Ric)$$

□

Corollary 6.1. *If* ∇^{LC} *is the Levi-Civita connection of an Einstein Riemannian manifold,then:*

$$J_{\nabla^{LC}} = \{0\}$$

Proposition 6.5. *Let* (M, g) *be a compact Riemanian manifold such that the*

$$Ric < 0$$

then:

$$J_{\nabla^{LC}} = \{0\}$$

Proof. By using the formula

$$L_X\nabla^{LC} = i_X R^{\nabla^{LC}} + \nabla^{LC2}X$$

we deduce that $J_{\nabla^{LC}}$ is subset of killing vector field on M.By assumption $Ric < 0$, From the theorem of Bockner we deduce that

$$J_{\nabla^{LC}} = \{0\}$$

□

As an easy consequence of Lie's third theorem, it comes:
Up to a Lie group isomorphism, it exists a unique simply connected Lie group G_∇ whose Lie algebra is isomorphic to the Lie algebra J_∇.

Definition 6.4. *[16] J_∇ is called completely integrable if its the linear counterpart of a locally effective differentiable acion:*

$$G_\nabla \times M \ni (\gamma, x) \to \gamma.x \in M.$$

In a compact manifold M every infinitesimal action of a finite dimensional Lie group is integrable. That is due to the fact every vector field X is complete in the meaning that X is a generator of a one parameter subgroup of the group of diffeomorphisms

7. A canonical representation of G_∇

Definition 7.1. *W be a finite dimensional real vector space. Its group of affine isomorphisms, denoted by Aff(W), is defined as the semi-direct product:*

$$Aff(W) = GL(W) \ltimes W$$

where W is by abuse of notation the group of translations of W.

There is a natural affine representation of the Lie algebra J_∇ in itself as a vector space:

$$J_\nabla \ni X \to \rho(X) = (\nabla_X, X) \in gl(J_\nabla) \times J_\nabla = aff(J_\nabla).$$

with affine action given by:

$$\rho(X).Y = \nabla_X Y + X, \quad \forall Y \in J_\nabla.$$

By virtue of the universal property of simply connected finite dimensional Lie groups, there exist a unique continuous affine representation

$$\gamma \in G_\nabla \to (f_\nabla(\gamma), q_\nabla(\gamma)) \in Aff(J_\nabla).$$

Proposition 7.1. *With the above notations, f_∇ is a linear representation of G_∇ in J_∇ and q_∇ is a J_∇ valued 1-cocycle of f*

Proof. The couple (f,q) is a continuous homomorphism of the Lie group G_∇ on the Lie group $Aff(J_\nabla)$, thus, for any $\gamma_1, \gamma_2 \in G_\nabla$:

$$(f(\gamma_1).f(\gamma_2), f(\gamma_1)q(\gamma_2) + q(\gamma_1)) = (f(\gamma_1.\gamma_2), q(\gamma_1.\gamma_2)).$$

□

The cohomology of the Lie group G_∇ value in his Lie algebra J_∇ is defined by the complex [5–7] :

$$... \to C^q(G_\nabla, J_\nabla) \to C^{q+1}(G_\nabla, J_\nabla) \to C^{q+2}(G_\nabla, J_\nabla) \to$$

with differential operator D:

$$D\theta(\gamma_1, ..., \gamma_{q+1}) = f_\nabla(\gamma_1).\theta(\gamma_2, ..., \gamma_{q+1}) + \sum_{i \leqslant q}(-1)^i \theta(.., \gamma_i\gamma_{i+1}, ...) + (-1)^q \theta(\gamma_1, ..., \gamma_q)$$

The condition :

$$q_\nabla(\gamma_1.\gamma_2) = f_\nabla(\gamma_1)q_\nabla(\gamma_2) + q_\nabla(\gamma_1)$$
$$\forall \gamma_1, \gamma_2 \in G_\nabla$$

is equivalent to $q_\nabla \in Z^1(G_\nabla, J_\nabla)$. The next definition thus makes sens: The cohomology class $[q_\nabla] \in H^1(G_\nabla, J_\nabla)$ is called the radiant class of the affine representation (f, q). We are now in position to state one of the main results of the article:

Theorem 7.2. *The following statements are equivalent :*

(1) *The affine action*

$$G_\nabla \ltimes J_\nabla \ni (\gamma, X) \to f_\nabla(\gamma).X + q_\nabla(\gamma) \in J_\nabla$$

has a fixed point;

(2) *The cohomology class* $[q_\nabla]$ *vanishes;*

(3) *The affine representation*

$$G_\nabla \ni \gamma \to (f_\nabla(\gamma), q_\nabla(\gamma) \in Aff(J_\nabla)$$

is conjugated to the linear representation

$$G_\nabla \ni \gamma \to f_\nabla(\gamma) \in GL(J_\nabla).$$

Proof. Let us first show that (1) implies (2). Let $-Y_0$ be a fixed point of the affine action (f_∇, q_∇), then

$$f_\nabla(\gamma)(-Y_0) + q_\nabla(\gamma) = -Y_0, \quad \forall \gamma \in G_\nabla.$$

Therefore one has:

$$q_\nabla(\gamma) = f_\nabla(\gamma)(Y_0) - Y_0, \quad \forall \gamma \in G_\nabla.$$

So the cocycle q is exact. To prove that (2) implies (3), consider the affine isomorphism (e, Y_0). It is nothing but the translation by Y_0

$$X \to X + Y_0;$$

We calculate

$$(e, Y_0)\dot{(}f_\nabla(\gamma), q_\nabla(\gamma))\dot{(}e, Y_0)^{-1} = (f_\nabla(\gamma), 0_\nabla)$$

Where O_∇ stands for the zero element of the vector space J_∇. Finally, (3) implies (1). This assertion means that there exists an affine isomorphism

$$J_\nabla \ni Y \to L(Y) + X_0 \in J_\nabla$$

such that

$$(L, X_0)\dot{(}f_\nabla(\gamma), q_\nabla(\gamma))\dot{(}L, X_0)^{-1} = (f_\nabla(\gamma), 0_\nabla), \quad \forall \gamma \in G_\nabla$$

The calculation of the left member yields the following identities:

$$(a): \quad L\dot{f}_\nabla(\gamma)\dot{L} = f_\nabla(\gamma) \tag{7.1}$$

$$(b): \quad L(q_\nabla(\gamma)) + X_0 - [L\dot{(}f_\nabla(\gamma)\dot{L}^{-1}](X_0) = 0_\nabla. \tag{7.2}$$

The identity (b) yields:

$$q_\nabla(\gamma) = f_\nabla(\gamma)(L^{-1}(X_0)) - L^{-1}(X_0), \quad \forall \gamma \in G_\nabla.$$

Taking into account identity (a), we obtain:

$$q_\nabla(\gamma) = f_\nabla(\gamma)(X_0) - X_0, \quad \forall \gamma \in G_\nabla.$$

So the vector $-X_0$ is a fixed point of the affine representaion (f_∇, q_∇). □

Definition 7.3. *The affine representation* (f_∇, q_∇) *is called the canonical affine representation of the gauge structure* (M, ∇)

When the infinitesimal action J_∇ is integrable, the proposition above is a key tool to relate the canonical affine representaion of (M, ∇) and the hyperbolicity problem for the orbits of G_∇.

8. Application to statistical manifolds

8.1. **Statistical manifolds.** In this section, We restrict our attention to Riemannian statistical manifolds (excluding the pseudo-Riemannian case) whose particular cases are non-degenerate Fisher information metrics of statistical models and their family of α-connections, $\alpha \in \mathbb{R}$. We recall that a statistical manifold can be viewed as a triple (M, g, ∇) formed of a Riemannian manifold (M, g) and a symmetric gauge structure (M, ∇) which are linked by the following identity:

$$(\nabla_X g)(Y, Z) - (\nabla_Y g)(X, Z) = 0, \quad \forall (X, Y, Z) \subset \mathcal{X}(M)$$

The g-dual, $\nabla^\star$ is defined as:

$$g(\nabla^\star_X Y, Z) = Xg(Y, Z) - g(Y, \nabla_X Z), \quad \forall (X, Y, Z).$$

Depending on needs, we will use the alternative definition:

Definition 8.1. *A statistical manifold is a quadruple $(M, g, \nabla, \nabla^\star)$ formed a positive Riemannian manifold (M, g) and pair of symmetric gauge structure $(M, \nabla, \nabla^\star)$ which are linked by the following identity*

$$Xg(Y, Z) - g(\nabla_X Y, Z) - g(Y, \nabla^\star_X Z) = 0, \quad \forall (X, Y, Z).$$

Example 1: Any Riemannian manifold (M, g, ∇^{LC}) where ∇^{LC} is the Levi-Civita connection of g is a statistical manifold as ∇^{LC} is self-dual. We say in such a case that M is a trivial statistical manifold.

Example 2: Let (M, g, ∇, ∇^*) any statistical manifold, the family $(M, g, \nabla^{(\alpha)}, \nabla^{(-\alpha)})_{\alpha \in \mathbb{R}}$ where $\nabla^{(\alpha)} = \frac{1+\alpha}{2}\nabla + \frac{1-\alpha}{2}\nabla^*$ is a statistical manifold for any $\alpha \in \mathbb{R}$.

Example 3: Let (M, g) some Riemannian manifold with ∇^{LC} the Levi-Civita connection with respect to g, and let $E \in TM \setminus \{0\}$. The triplet (M, g, ∇, ∇^*) is a statistical manifold where $\nabla_X Y = \nabla^{LC}_X Y + g(X, E)g(Y, E)E$ and $\nabla^*_X Y = \nabla^{LC}_X Y - g(X, E)g(Y, E)E$.

8.2. **Hyperbolic leaves in statistical manifold.** Let (M, g, ∇, ∇^*) be a compact statistical manifold and ∇^{LC} be its Levi-Civita connection.We consider three gauge structures (M, ∇), $(M, \nabla^\star)$ and (M, ∇^{LC}).By assumption, M is compact,therefore J_∇, $J_{\nabla^\star}$ and $J_{\nabla^{LC}}$ are the infinitesimal counterpart of the following locally effective differentiable dynamical systems:

$$G_\nabla \times M \to M \tag{8.1}$$

$$G_{\nabla^\star} \times M \to M \tag{8.2}$$

$$G_{\nabla^{LC}} \times M \to M \tag{8.3}$$

Remark 8.2. *In the Riemannian geometry viewpoint, the orbits of $G_{\nabla^{LC}}$ are flat Riemannian manifolds. Up to an isometry and a finite covering, each such an n-dimensional orbit is a flat cylinder of over an Euclidean torus [1], namely*

$$(\frac{\mathbb{T}^k}{\Gamma} \times \mathbb{R}^{n-k}, g_0)$$

with the flat metric g_0 induced from the ambient Euclidean metric of $\mathbb{R}^n$. The integer k is the first Betti number of the orbit and Γ is finite group of isometry.

Notation 8.3. *$\mathcal{F}_\nabla$ is the foliation whose leaves are orbits of the Lie group G_∇ and g_∇ the restriction of g to $\mathcal{F}_\nabla$.*

Theorem 8.4. *Let $(M, g, \nabla, \nabla^\star)$ be a compact statistical manifold. Then the foliation $\mathcal{F}_\nabla$ (resp. $\mathcal{F}_{\nabla^*}$) is a Hessian foliation in (M, g, ∇) (resp. (M, g, ∇^*)).*

Proof. (1) By assumption M is compact then J_∇ is integrable,we conclude that J_∇ is an infinitesimal action of G_∇ on M.We have $R^\nabla(X, Y)J_\nabla = 0$ then $\mathcal{F}_\nabla$ is locally flat foliation in (M, ∇). The G_∇-orbits are ∇-auto-parallel then $\delta_{KV} g_\nabla = 0$. The leaves of $\mathcal{F}_\nabla$ are Hessian submanifolds of (M, g, ∇).
(2) By using the same arguments for $G_{\nabla^\star}$, the theorem is proved. □

Theorem 8.5. *In a compact statistical manifold $(M, g, \nabla, \nabla^\star)$ the following assertions are equivalent:*

(1) $[g_\nabla] = 0 \in H^2_{KV}(\mathcal{F}_\nabla)$

(2) $[q_{\nabla^\star}] = 0 \in H^1(G_{\nabla^\star}, J_{\nabla^\star})$

(3) $(f_{\nabla^\star}, q_{\nabla^\star})$ *has a fixed point.*

(4) $(f_{\nabla^\star}, q_{\nabla^\star})$ *is affinely conjugated to its linear component $f_{\nabla^\star}$.*

Proof. According to Theorem 7.2, assertions (2), (3) and (4) are equivalent. Therefore, it is sufficient to prove that assertions (1) and (2) are equivalent.
Let us demonstrate first that (1) implies (2)

Since the class $[g_\nabla]$ vanishes, it exist a de Rham closed differential 1-form θ such that:

$$g_\nabla(X, Y) = -X\theta(Y) + \theta(\nabla_X Y), \quad \forall(X, Y).$$

By the defining property of statistical manifolds, it comes:

$$Xg_\nabla(Y, Z) = g_\nabla(\nabla^\star_X Y, Z) + g_\nabla(Y, \nabla_X Z), \quad \forall(X, Y, Z).$$

Let H be the unique vector field satisfying:

$$\theta(X) = g_\nabla(H, X), \quad \forall X.$$

Using once again the defining property of statistical manifolds, we get:

$$Xg_\nabla(H, Y) - g_\nabla(\nabla^\star_X H, Y) - g_\nabla(H, \nabla_X Y) = 0.$$

Since left hand member is $C^\infty(M)$-multilinear we can assume that $H \in J_\nabla$ and we get the following identity

$$g_\nabla(\nabla^\star_X H, Y) = Xg_\nabla(H, Y) - g_\nabla(H, \nabla_X Y) = g_\nabla(X, Y), \quad \forall(X, Y) \subset \mathcal{X}(M).$$

Thus one has:

$$\nabla^\star_X(-H) - X = 0_\nabla, \quad \forall X \in \mathcal{X}(M).$$

So the $-H$ is a fixed point of $(f_{\nabla^\star}, q_{\nabla^\star})$.
Let us demonstrate now that (2) implies (1).

Let us assume that $(f_{\nabla^\star}, q_{\nabla^\star})$ has a fixed point $Y_0 \in J_{\nabla^\star}$. Then:

$$f_{\nabla^\star}(\gamma)(Y_0) + q_{\nabla^\star}(\gamma) = Y_0, \quad \forall \gamma \in G_{\nabla^\star}.$$

To every $X \in J_{\nabla^\star}$, we assign the one parameter subgroup

$$\{Exp(tX), t \in \mathbb{R}\} \subset G_{\nabla^\star}.$$

We have:

$$f_{\nabla^\star}(Exp(tX))(Y_0) + q_{\nabla^\star}(Exp(tX)) = Y_0, \quad \forall t \in \mathbb{R}.$$

Calculating the derivative at $t = 0$, one obtains:

$$\nabla^\star_X Y_0 + X = 0.$$

Finally:

$$X g_\nabla(Y_0, Y) = g_\nabla(\nabla^\star_X Y_0, Y) + g_\nabla(Y_0, \nabla_X Y)$$

Using $\nabla^\star_X Y_0 = -X, \forall X$, one obtains the following identity:

$$X g_\nabla(Y_0, Y) = -g_\nabla(X, Y) + g_\nabla(Y_0, \nabla_X Y).$$

By putting $\theta(Y) = -g_\nabla(Y_0, Y)$, it comes:

$$g_\nabla(X, Y) = X\theta(Y) - \theta(\nabla_X Y), \quad \forall (X, Y)$$

Then one has $[g_\nabla] = 0 \in H^2_{KV}(\nabla)$, concluding the proof. □

Corollary 8.1. *Let $(M, g, \nabla, \nabla^\star)$ such that both J_∇ and $J_{\nabla^\star}$ be integrable. We assume that leaves of both $\mathcal{F}_\nabla$ and $\mathcal{F}_{\nabla^\star}$ satisfy one among the assertions of Theorem 8.5. Then every compact leaf of $\mathcal{F}_\nabla$ (resp. $\mathcal{F}_{\nabla^\star}$) is hyperbolic.*

8.3. **Cohomological characteristical of hyperbolicity of Hessian manifolds.** The following nullity of Hessian class,nullity of radiant class affine representation and hyperbolicity of locally flat manifold the following theorem is another formulation of the hyperbolicity of a locally flat manifold in the sense of Koszul. It proves that the hyperbolicity can be described by the nullity of the radiant class of the affine dynamics of the dual connection.

Theorem 8.6. *Let (M, g, ∇) being a compact Hessian manifold.If the characteristic obstruction*

$$\{[g_\nabla] \cup [q_{\nabla^\star}]\} = \{0\} \cup \{0\}$$

Then

$$(M, \nabla) \textit{ is an hyperbolic manifold}$$

.

Theorem 8.7 ([20], [10], [23]). *Let M be a closed hyperbolic manifold.Then M is a quotient of a properly convex cone.*

Theorem 8.8. *Let (M, ∇) be a closed hyperbolic manifold.Then*

$$[q_{\nabla^\star}] = 0 \in H^1(G_{\nabla^\star}, J_{\nabla^\star})$$

and

$$[q_\nabla] = 0 \in H^1(G_\nabla, J_\nabla)$$

Proof. Step 1: Koszul [13]
From theorem 9.7, the universal covering $\tilde{M}$ is diffeomorphic to a convex cone not containing any straight line of $\mathbb{R}^n$,let's call it by Ω and q a affine map from Ω to M define by $p = \pi \circ \mathcal{D}^{-1}$,where $\pi : \tilde{M} \longrightarrow M$ and the developing map $\mathcal{D} : \tilde{M} \longrightarrow \Omega$.Let $H^{'} \in \mathcal{X}(\mathbb{R}^n)$ define by $H^{'}(x) = \sum_{i=1}^{n} x^i \frac{\partial}{\partial x^i}$ is homotheties fields (ie this field is generated by a 1-parameter group of homotheties of center 0 and ratio e^t,then for all $X^{'} \in \mathcal{X}(\mathbb{R}^n)$,we have $\nabla^0_{X'} H^{'} = X^{'}$ where ∇^0 is canonical flat connection on $\mathbb{R}^n$.The vector field $H^{'}$ is $Gl(\mathbb{R}^n)$-invariant.Since Ω is salient then any affine automorphism of Ω is restriction of an element of $Gl(\mathbb{R}^n)$.The restriction of $H^{'}$ to Ω is invariant by the affine automorphisms of Ω. Therefore, there exists one and only one vector field H on M, such that $H^{op} = H'$.Since p is locally an isomorphism of locally flat manifolds ie($dp(\nabla^0) = \nabla$):

$$\nabla_X H = \nabla^0_{X'} H^{'}$$

$$\nabla_X H = X$$

then $[q_\nabla] = 0 \in H^1(G_\nabla, J_\nabla)$
step 2
By assumption (M, ∇) is compact hyperbolic, so it exists a closed differentiable one form α such that $\nabla\alpha$ is positive define. Then $g = \nabla\alpha$ is Hessian metric and locally $g_{ij} = \frac{\partial^2\phi}{\partial i \partial j}$.We thus conclude that (M, g, ∇) is a Hessian manifold. Let $\nabla^\star$ be the dual of ∇, defined by:

$$g(\nabla^\star_X Y, Z) = X.g(Y, Z) - g(Y, \nabla_X Z)$$

$\nabla^\star$ is a locally flat connection on M.

We have:

$$g(X, Y) = X.\alpha(Y) - \alpha(\nabla_X Y)$$

thus, it exists $H^\star \in \mathcal{X}(M)$ such that $\alpha(X) = g(H^\star, X)$,then we conclude that

$$g(X, Y) = g(\nabla^\star_X H^\star, Y)$$

for all $Y \in \mathcal{X}(M)$. We conclude that:

$$\nabla^\star_X H^\star = X$$

then

$$[q_{\nabla^\star}] = 0 \in H^1(G_{\nabla^\star}, J_{\nabla^\star})$$

□

Corollary 8.2. *Let* (M, ∇) *be a closed hyperbolic manifold.Then there exists closed one form* Θ *such that* $g = \nabla^\star\Theta$.

Theorem 8.9. *Let* (M, ∇) *be a closed hyperbolic manifold.Then* $(M, \nabla^\star)$*is also hyperbolic manifold.*

9. When the orbits are quotients of convex cones.

Convex cones are examples of bounded domains. The studies of convex cones have been and continue to be among high standing subjects in geometry and in analysis. The pioneering works are those of Elie Cartan, but there is a wealth of subsequent works, see cartan1935domaines, koszul1961domaines, vinberg1967theory and many others. Nowadays the analysis in convex cones plays interesting role in the information geometry, barbaresco2013information. Theorem 3.3 provides conditions under which compact orbits of G_∇ are hyperbolic. We are going to demonstrate that up to difeomorphisms, compact leaves of $\mathcal{F}_\nabla$ are quotient homogeneous convex cones. According to our previous notation, if M is a compact leaf of $\mathcal{F}_\nabla$ then $Q(\tilde{M})$ is a homogeneous convex cone. For convenience, we introduce the following notion:

Let $(N, g, \nabla, \nabla^\star)$ be an integrable statistical manifold whose Hessian foliations are denoted by

$$\{\mathcal{F}_\nabla, g_\nabla, \nabla\}$$

and

$$\left\{\mathcal{F}_{\nabla^\star}, g_{\nabla^\star}, \nabla^\star\right\}.$$

Henceforth, by abuse of notation, a leaf of

$$\{\mathcal{F}_\nabla, g_\nabla, \nabla\}$$

is an Hessian manifold that we denote by

$$(M, g_\nabla, \nabla).$$

Its g-dual is:

$$(M, g_\nabla, \nabla^\star).$$

Theorem 9.5 links the Hessian class g_∇, namely:

$$[g_\nabla] \in H^2_{KV}(\nabla)$$

and the radiant class of $(M, g, \nabla^\star)$, namely:

$$[q_{\nabla^\star}] \in H^1(G_{\nabla^\star}, J_{\nabla^\star}).$$

We deduce that if M is compact then:

$$[g_\nabla] \cup [q_{\nabla^\star}]$$

is a characteristic obstruction of (M, ∇) being hyperbolic. Henceforth we assume these obstructions vanish.

We have already pointed out that if $\tilde{M}$ the universal covering of M, it admits a unique locally flat structure $(\tilde{M}, \tilde{\nabla})$ such that the covering map

$$\pi : \tilde{M} \to M$$

is a gauge morphism between $\tilde{\nabla}$ and ∇.

Theorem 9.1. *[11] Let*

$$\{(M, g, \nabla), \quad [g_\nabla] \cup [q_{\nabla^\star}]\}$$

being a compact Hessian manifold. If:

$$\{[g_\nabla] \cup [q_{\nabla^\star}]\} = \{0\} \cup \{0\}.$$

Then for $Q(\tilde{M})$ being a cone, it is sufficient that $\tilde{M}$ being homogeneous under the group of transformation of $(\tilde{M}, \tilde{\nabla})$♣

Theorem 9.2. *Let (N, g, ∇) be a statistical manifold and let $\mathcal{F}_\nabla$ be its canonical Hessian foliation. If the characteristic obstruction vanishes:*

$$\{[g_\nabla] \cup [q_{\nabla^\star}]\} = \{0\} \cup \{0\}.$$

Then, up to a affine diffeomorphism, every compact leaf of $\mathcal{F}_\nabla$ is the quotient of a sharp open convex cone by a discrete group of $GL(\mathbb{R}^n)$

♣

Proof. The action

$$G_\nabla \times M \to M$$

gives rise to the action

$$G_\nabla \times \tilde{M} \to \tilde{M}.$$

(M, ∇) is homogeneous under the action of the Lie group G_∇ and (M, ∇) is the quotient of $(\tilde{M}, \tilde{\nabla})$ under the action of the fundamental group $\pi_1(M)$.
Therefore both $\pi_1(M)$ and G_∇ are subgroups of the group

$$Aff(\tilde{M}, \tilde{\nabla}).$$

It is easy to check that G_∇ is included in the normalizer of $\pi_1(M)$.
Since M is transitively acted on by G_∇, every orbit of G_∇ in $\tilde{M}$ is an open submanifold of $\tilde{M}$. Since $\tilde{M}$ is connected the action of G_∇ on $\tilde{M}$ is transitive.From the theorem 10.2 we deduce that the compact leaves of $\mathcal{F}_\nabla$ are diffeomorphic to the quotient of a sharp open convex cone by a discrete group of $GL(\mathbb{R}^n)$. □

10. Statistical models on the leaves of $\mathcal{F}_\nabla$

10.1. **Characteristic function and Formalism of Barbaresco.** The locally flat structure of (M, ∇) rises up on its universal covering $\tilde{M}$ as a locally flat structure $(\tilde{M}, \tilde{\nabla})$ for which the projection $\pi : (\tilde{M}, \tilde{\nabla}) \longrightarrow (M, \nabla)$ is affine map.The locally flat structure on $\tilde{M}$ is defined by the local affine diffeomorphism $\mathcal{D} : (\tilde{M}, \tilde{\nabla}) \longrightarrow (\mathbb{R}^n, \nabla^0)$ where ∇^0 is the standard locally flat connection define by $\nabla^0 \frac{\partial}{\partial xi} = 0 \; \forall i = 1, 2, ..., n$.
The development map gives rise to an representation $H(\nabla)$ of the fundamental group $\pi_1(M)$ of M in $Aff(\mathbb{R}^n)$, called the holonomy representation of M. It is defined by the following commutative diagram:

$$\begin{array}{ccc} (M, \tilde{\nabla}) & \xrightarrow{\pi_1(M)} & (\tilde{M}, \tilde{\nabla}) \\ \downarrow \mathcal{D} & & \downarrow \mathcal{D} \\ (\mathbb{R}^n, \nabla^0) & \xrightarrow{H(\nabla)} & (\mathbb{R}^n, \nabla^0) \end{array}$$

$$\mathcal{D} \circ \gamma = H(\nabla) \circ \mathcal{D}$$

$$< \tilde{M}, \tilde{\nabla}, \pi_1(M) > \longrightarrow (\Omega, \nabla^0, H(\nabla)) \subset (\mathbb{R}^n, \nabla^0)$$

By the theorem 53,Ω is sharp open convex cone of $\mathbb{R}^n$ and $H(\nabla) \subset Gl(\mathbb{R}^n)$, thus it exists a local diffeomorphism $\tilde{D}$ fitting in the next diagram:

$$\begin{array}{ccc} <\tilde{M}, \tilde{\nabla}, \pi_1(M)> & \xrightarrow{D} & (\Omega, \nabla^0, H(\nabla)) \\ \downarrow \pi & & \downarrow \tilde{\pi} \\ (M, \nabla) & \xrightarrow{\tilde{D}} & (\frac{\Omega}{H(\nabla)}, \bar{\nabla}^0) \end{array}$$

Let Ω sharp open convex cone of $\mathbb{R}^n$ and let $\bar{\Omega}$ the closure of Ω. The set:

$$\Omega^\star = \left\{\Psi \mid (\Psi, x) > 0, \forall x \in \bar{\Omega} - \{0\}\right\}$$

is called the dual cone of Ω.The holonomy $H(\nabla)$ act on $\Omega^\star$by dual representation

$$H(\nabla) : \Omega^\star \ni \xi \longrightarrow \tilde{\gamma}.\xi = \xi \circ \gamma^{-1} \in \Omega^\star$$

where $\gamma \in H(\nabla)$ J-L Koszul and E.B. Vinberg have introduced the following integral ρ of a sharp convex cone Ω:

$$\rho_\Omega(\theta) = \int_{\Omega^\star} e^{-<\theta,\xi>} d\xi, \forall \theta \in \Omega \tag{10.1}$$

where $d\xi$ is the Lebesgue measure of $\mathbb{R}^\star$.
ρ is an analytic function on Ω,with $\rho_\Omega(\theta) \in]0, +\infty[$,called the Koszul-Vinberg characteristic function of cone Ω. It satisfies [21]:

(1) ρ_Ω is logarithmically strictly convex and $\Phi_\Omega(\theta) = log(\rho_\Omega)$ is strictly convex.

(2) If $\gamma \in H(\nabla)$ then $\rho_\Omega(\gamma\theta) = det(\gamma)^{-1}\rho_\Omega(\theta)$.

(3) Koszul 1-form:The differential 1-form $\alpha = dlog\rho_\Omega = \frac{d\rho_\Omega}{\rho_\Omega}$ is invariant by $H(\nabla)$. If $\theta \in \Omega$ and $u \in \mathbb{R}^n$ then $<\alpha_\theta, u> = -\int_{\Omega^\star} <\xi, u> e^{-<\xi,\theta>} d\xi$ and $\alpha_\theta \in -\Omega^\star$.

(4) Koszul 2-form β :The symmetric differential 2-form $\beta = \nabla\alpha = d^2log(\rho_\Omega)$ is positive definite symmetric bilinear form on $\mathbb{R}^n$ invariant under $H(\nabla)$.$\nabla\alpha > 0$(Schwarz inequality and $d^2log\rho_\Omega(u, v) = \int_{\Omega^\star} <\xi, u><\xi, v> e^{-<\xi,u>} d\xi)$

(5) Koszul-Vinberg Metric: The Riemanian metric = $d^2log\rho_\Omega$ is invariant by $H(\nabla)$.

In [2],Frédéric Barbaresco introduced a probability density called Koszul density and defined by:
For $\theta \in \Omega$ and $\xi \in \Omega^\star$,we define

$$P(\theta, \xi) = \frac{e^{-<\theta,\xi>}}{\rho_\Omega} = exp\left\{- <\theta, \xi> -log(\rho(\theta))\right\}.$$

Then $\{P(\theta, \xi), \xi \in \Omega^\star\}$ is an exponential family of probability distributions on $\Omega^\star$ parameterized by $\theta \in \Omega$.

10.2. **Construction of Probability density on leaves of $\mathcal{F}_\nabla$.** Our aim is to provide the leaves of $\mathcal{F}_\nabla$ with a statistical model structure.
$H(\nabla) \subset Gl(n, \mathbb{R})$ is an amenable group, thus it exists a mean $\mu \in Hom(L^\infty(H(\nabla)), \mathbb{R})$ defined by:

$$\mu(f) = \int_{H(\nabla)} f(\gamma) d\mu(\gamma)$$

and satisfying:

(1) If $f > 0$, $\mu(f) > 0$.
(2) $\mu(f \circ \gamma) = \mu(f) \forall f, \forall \gamma \in H(\nabla)$.
(3) $\mu(1) = 1$.

Let:

$$f_{\theta,\xi} : \gamma \ni H(\nabla) \longrightarrow P(\theta\gamma, \xi) = P(\theta, \gamma^t \xi) \in \mathbb{R}$$

we have:

$$P(\theta\gamma, \xi) \leq 1$$

Then:

$$f_{\theta,\xi} \in L^\infty(H(\nabla))$$

Proposition 10.1.

$$Q(\theta, \xi) = \mu(P_{\theta,\xi}) = \int_{H(\nabla)} P(\theta\gamma, \xi) d\mu(\gamma)$$

is $H(\nabla)$-invariant ie:

$$Q(\gamma\theta, \xi) = Q(\theta, \xi)$$

Proposition 10.2. *Let Q the map*

$$Q : \frac{\Omega}{H(\nabla)} \times \Omega^\star \longrightarrow \mathbb{R}$$

we have

$$\int_{\Omega^\star} Q(\theta, \xi) d\xi = 1$$

Proof.

$$\int_{\Omega^\star} Q(\theta, \xi) d\xi = \int_{\Omega^\star} [\int_{H(\nabla)} P(\theta\gamma, \xi) d\mu(\gamma)] d\xi$$

By the Theorem of Fubini-Tonelli,we have

$$\int_{\Omega^\star} Q(\theta, \xi) d\xi = \int_{\Omega^\star \times H(\nabla)} P(\theta\gamma, \xi) \tilde{d}(\mu, \xi)$$

$$\int_{\Omega^\star} Q(\theta, \xi) d\xi = \int_{H(\nabla)} [\int_{\Omega^\star} P(\theta\gamma, \xi) d\xi] d\mu(\gamma)$$

$$\int_{\Omega^\star} Q(\theta, \xi) d\xi = \int_{H(\nabla)} 1 d\mu(\gamma)$$

$$\int_{\Omega^\star} Q(\theta, \xi) d\xi = 1$$

□

Theorem 10.1. *Let (N, g, ∇) be an integrable statistical manifold and let $\mathcal{F}_\nabla$ be its canonical Hessian foliation. If the characteristic obstruction vanishes:*

$$\{[g_\nabla] \cup [q_{\nabla^\star}]\} = \{0\} \cup \{0\}.$$

Then, up to an affine diffeomorphism, every compact leaf of $\mathcal{F}_\nabla$ is a parametric space of statistical models for a measurable set $(\Omega^\star, \mathcal{B}(\Omega^\star))$

$$< M, Q >$$

Where Q is define by:

$$Q(\Theta, \xi) = \int_{H(\nabla)} \frac{e^{-<\gamma\Theta, \xi>}}{\int_{\Omega^\star} e^{-<\gamma\Theta,\xi>} d\xi} d\mu^{H(\nabla)}$$

with $H(\nabla) \subset GL(\mathbb{R}^n)$

11. A worked example

We shall give a example of compact hessian manifold whose orbits are quotient of statistical models by a discrete subgroup.Let $\mathbb{R}^2$ be a 2-dimensional real affine space with the natural flat affine connection D and let $\{x, y\}$ be an affine coordinate system of $\mathbb{R}^2$.
Let Ω be a domain define by $x > 0$ and $y > 0$.Consider a Riemannian metric on Ω given by $g = \frac{1}{x^2} dx^2 + \frac{1}{y^2} dy^2$.
Then (D,g) is Hessian structure on Ω.
Let Σ and χ be linear transformations on Ω defined by :

$$\Sigma : (x, y) \to (2x, y)$$

$$\chi : (x, y) \to (2x, 3y)$$

Then $<\Sigma,\chi>$ leave the Hessian structure (D, g) invariant.
We denote Γ the group generate by $\{\Sigma, \chi\}$, we can also write Γ as $< \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} ; \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} >$.
Γ acts properly discontinuously on Ω and $\frac{\Omega}{\Gamma}$ is is compact Hessian manifold which is diffeomorphic to a Torus.
Let us denote by π the projection from Ω to $\frac{\Omega}{\Gamma}$ and by (D, g) the Hessian structure on $\frac{\Omega}{\Gamma}$.Since the space of all Γ-invariant D parallel 1-forms on $\Omega$is spanned by dx and dy.The space of all D-parallel 1-forms on $\frac{\Omega}{\Gamma}$ is spanned by ω and Φ where $dx = \pi^\star\omega$ and $dy = \pi^\star\Phi$.Let X and $\tilde{X}$ be a vector field on $\frac{\Omega}{\Gamma}$ defined by $\omega(Y) = g(X, Y)$ and $\Phi(Y) = g(\tilde{X}, Y)$for each vector field Y on $\frac{\Omega}{\Gamma}$.Then $X = \pi_\star(\frac{\partial}{\partial x})$ and $\tilde{X} = \pi_\star(\frac{\partial}{\partial y})$ and the vector space J_∇ of all Hessian vector fields on $\frac{\Omega}{\Gamma}$ is spanned by $<X,\tilde{X}>$.Since T^2 is compact the J_∇ is integrable.Let $Exp(tX)$ and $Exp(tY)$ is a 1-parameter group of transformations generated by X and by Y.Let $G_\nabla =< Exp(tX), Exp(tY) >$. Consider $a > 0$ and $b > 0$, the compact homogeneous the orbit $G_\nabla\pi(a, b) = \{\gamma.\pi(a, b), \gamma \in G_\nabla\}$ is a circle. Then we conclude that (S^1, D, g_{S^1}) is Hessian manifold and satisfy $D\frac{\partial}{\partial\theta} = 0$ and $g_{S^1} = \frac{1}{\theta^2} d\theta^{\otimes 2}$.By using conjugation we have $D^\star_{\frac{\partial}{\partial\theta}} \frac{\partial}{\partial\theta} = -\frac{2}{\theta} \frac{\partial}{\partial\theta}$.Let $H^\star = h(\theta)\frac{\partial}{\partial\theta}$ vectors fields on S^1.$H^\star$ is homothety vector fields of $D^\star$ if h is solution of differential equation $y^{'} - \frac{2}{\theta^2} y = 1$.We deduce that

$$h(\theta) = Ae^{\frac{-2}{\theta}} + 2e^{\frac{-2}{\theta}} \int_{\frac{-2}{\theta}}^{+\infty} \frac{e^{-s}}{s} \mathrm{d}s + \theta$$

where A is constant.
As $D^{\star}H^{\star} = Id$ then (S^1, D) is hyperbolic.Now let $p : \mathbb{R} \to S^1$ the covering map, and consider the map

$$(\mathbb{R}, \tilde{D}) \to (S^1, D)$$

.The compacts orbits is $G_{\nabla}\pi(a,b)$ is quotient $\frac{\mathbb{R}^{>0}}{\Gamma}$, here Γ is discrete subgroup of $Gl(\mathbb{R})$

12. Conclusion and future works

The characterization of hyperbolicity given in theorem 8.5 has an important application in elucidating the structure of some statistical manifolds as statistical models. In a future work, an explicit construction of an exponential family on leaves of a statistical manifold will be given along with an application to the classification of compact exponential models in dimension 4.

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