NON DEGENERACY OF AFFINELIKE LIE ALGEBRA

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Abstract. The aim of this note is to prove some cohomological vanishing theorems for non solvable affinelike Lie algebras, say ALLA. There are some relevant consequences of our vanishing theorems:

(1) Every real non solvable affinelike Lie algebra is formally nondegenerate in the sense of A. Weinstein, (Theorem 1.1).
(2) Let G be a non solvable Lie group whose Lie algebra is an affinelike Lie algebra (g,e). If the radical of [ker(ad(e)),ker(ad(e))] is commutative, then G admits a left invariant symplectic structure if and only if it has an open coadjoint orbit, (Theorem 1.2).

(a) In Section 6, we use our vanishing theorems to supply an algebraic proof of the normal form theorem for Lie non solvable a-algebroids.

1. Introduction

Let Π be a smooth Poisson tensor on a smooth manifold M and let:

$$A_\Pi = (T^*M, \natural)$$

be the associated Lie algebroid structure in the cotangent bundle. The anchor map $\natural$ is the vector bundle homomorphism from T^*M to the tangent bundle TM defined by:

$$\natural(\alpha) = i(\alpha)\Pi.$$

Both Π and $\natural$ have the same singularities. Let x_0 be a singular point of Π. Let us denote by $\tilde{\Pi}$ (resp. $\tilde{A_\Pi}$) the Taylor expansion at x_0 of π (resp. $\natural$). Both $\tilde{\Pi}$ and $\tilde{A_\Pi}$ have the same linear part, namely Π^1. This linear part defines a Lie algebra structure in the vector space $T^*_{x_0}M$. We denote this Lie algebra structure by $(T^*_{x_0}M, \Pi^1)$. An important and largely open question in the Poisson geometry is to determinate those Lie algebras which are non degenerate in the sense of A. Weinstein. If a Lie algebra $\mathfrak{g}$ is formally (resp. analytically or smoothly) non degenerate in the sense of A. Weinstein, then, for every formal (resp. analytic or smooth) Poisson tensor Π the equality:

$$(T^*_{x_0}M, \Pi^1) = \mathfrak{g}$$

implies the formal (resp. analytic or smooth) linearization (near x_0) of the Poisson tensor Π. Recently, Jean-Paul Dufour and Nguyen Tien Zung have proved that for every positive integer m, the affine Lie algebra

$$\mathrm{Aff}(m) = \mathfrak{gl}(m) \ltimes \mathbb{R}^m$$

is analytically non degenerate in the sense of Weinstein, [10]. The list of known types of non degenerate Lie algebras is short, (see [DUF-MOL], [DUFNGU)]). The linearization of Π

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implies that A_Π, is linearizable as well. The inverse generally fails to hold. However, Jean-Paul Dufour conjectures that the formal linearization would imply the analytic linearization of Poisson structures whose ranks are > 2, [DUF3]. In the analytic category, the formal linearization of Π implies its analytic linearization whenever the linear part $(T^*_{x_0}M, \Pi^1)$ is a semi-simple Lie algebra, [7, 8]. The formal linearization problem may be considered from two points of view. The first one is the splitting problem of extensions of Lie algebras, [21]. The second viewpoint is that of formal deformations of Lie algebra structures. The both viewpoints are largely governed by the Chevalley-Eilenberg cohomology of Lie algebras, [11, 20]. The concern of this work is the study of the formal nondegeneracy of a new class of Lie algebras, say affinelike Lie algebras, (or ALLA). Such an algebra $\mathfrak{g}$ contains an element e such that $\mathrm{ad}(e)$ satisfies:

$$\mathrm{ad}^2(e) = \mathrm{ad}(e)$$

Let us represent an affinelike Lie algebra by the couple $(\mathfrak{g}, e)$. From the Poisson geometry viewpoint, central results of this work yield the following statements:

Theorem 1. *Every non solvable affinelike Lie algebra is formally non degenerate in the sense of A. Weinstein. A connected Lie group G whose Lie algebra $\mathfrak{g}$ is an affinelike Lie algebra, namely $(\mathfrak{g}, e)$, is labelled affinelike Lie group.*

Theorem 2. *Let: $(\mathfrak{g}, e)$ be the Lie algebra of a connected non solvable affinelike Lie group G. Let us set $\mathfrak{g}_e = \ker(\mathrm{ad}(e))$ and, $\Delta\mathfrak{g}_e = [\mathfrak{g}_e, \mathfrak{g}_e]$. Suppose the radical of $\Delta\mathfrak{g}_e$ to be commutative. If G carries a left invariant symplectic structure, then it has an open coadjoint orbit in the dual vector space of its Lie algebra. Moreover, if the ground field is the field of complex numbers, then the open coadjoint orbit is unique and everywhere dense*

2. Affinelike Lie algebras

Let $\mathbb{K}$ be either the field $\mathbb{R}$ of real numbers or the field $\mathbb{C}$ of complex numbers. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{K}$. Given $e \in \mathfrak{g}$ we shall set:

$$\begin{cases} \Delta\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \\ \Delta_e\mathfrak{g} = [e, \mathfrak{g}] \\ \mathfrak{g}_e = \ker(\mathrm{ad}(e)) \end{cases}$$

Let $\mathfrak{S}$ be a Levi subalgebra of $\Delta\mathfrak{g}_e$. In the sequel, $R(\mathfrak{g})$ will be the radical of the Lie algebra $\mathfrak{g}$.

Definition 1. A Lie algebra $\mathfrak{g}$ is called affinelike Lie algebra, 'ALLA', if it contains an element e such that:

(alla 1) $\mathrm{ad}^2(e) = \mathrm{ad}(e)$.

(alla 2) $\mathfrak{g} = \mathbb{K}e \oplus \Delta g$.

(alla 3) If $\mathfrak{g}$ is non solvable, then $H^0(\mathfrak{S}, R(\Delta g)) = 0$.

According to property (alla 2) above, $\mathfrak{g}$ is a semi-direct product of $\Delta\mathfrak{g}$ and $\mathbb{K}e$. Here are two elementary properties of affinelike Lie algebras.

Proposition 1. *The subspace $\Delta_e\mathfrak{g}$ is a commutative ideal of $\mathfrak{g}$.*

Proof. Let $a, b \in \Delta_e\mathfrak{g}$. Then:

$$[e, [a, b]] = 2[a, b].$$

On the other hand one has:

$$[e, [e, [a, b]]] = [e, [a, b]].$$

Combining the two equalities above one easily sees that:

$$[a, b] = 0,$$

proving that $\Delta_e\mathfrak{g}$ is commutative. To see that it is an ideal of $\mathfrak{g}$, we remark that as a vector space, $\mathfrak{g}$ admits the following decomposition:

$$\mathfrak{g} = \Delta_e\mathfrak{g} \oplus \mathfrak{g}_e$$

Now given $x \in \mathfrak{g}$ let us write it as:

$$x = u + v,\ u \in \Delta_e\mathfrak{g}, v \in \mathfrak{g}_e$$

Given $a \in \Delta_e\mathfrak{g}$, one has:

$$[e, [x, a]] = [e, [u, a]] = [u, [e, a]] = [u, a] = [x, a].$$

proving that $\Delta_e\mathfrak{g}$ is an ideal of $\mathfrak{g}$. □

Proposition 2. *Since* $\mathrm{ad}(e)$ *is a derivation of the Lie algebra* $\mathfrak{g}$*, the subspace* $\mathfrak{g}_e$ *is a subalgebra of* $\mathfrak{g}$.

As a Lie algebra, $\mathfrak{g}$ may be presented as the semi-direct product:

$$\mathfrak{g} = \mathfrak{g}_e \ltimes \Delta_e\mathfrak{g}.$$

Putting:

$$\tilde{\mathfrak{g}}_e = \mathfrak{g}/\Delta_e\mathfrak{g}$$

we have an exact split sequence of Lie albegras:

$$0 \longmapsto \Delta_e\mathfrak{g} \longmapsto \mathfrak{g} \longmapsto \tilde{\mathfrak{g}}_e \longmapsto 0 \tag{1}$$

Remark 1. Under the adjoint representation of $\mathfrak{g}$ in itself, 1 is an exact sequence of $\mathfrak{g}$-modules which is generally not split. However, it is split as a sequence of $\mathfrak{g}_e$-modules.

Now, let W be a finite dimensional $\mathfrak{g}$-module. From 1 we deduce the following exact sequence of $\mathfrak{g}$-modules:

$$0 \longmapsto \mathrm{Hom}\,(W, \Delta_e\mathfrak{g}) \longmapsto \mathrm{Hom}\,(W, \mathfrak{g}) \longmapsto \mathrm{Hom}\,(W, \tilde{\mathfrak{g}}_e) \longmapsto 0 \tag{2}$$

As an exact sequence of $\mathfrak{g}_e$-modules, 2 is split. Let us denote by $C^*(\mathfrak{g}, \mathrm{Hom}\,(W, \mathfrak{g}))$ the Chevalley-Eilenberg complex of $\mathfrak{g}$ with coefficients in $\mathrm{Hom}\,(W, \mathfrak{g})$, [6]. Then, for every non negative integer k, we denote by $\mathrm{H}^k\,(\mathfrak{g}, \mathrm{Hom}\,(W, \mathfrak{g}))$ the k-th cohomology space of $\mathrm{H}\,(\mathfrak{g}, \mathrm{Hom}\,(W, \mathfrak{g}))$. From Remark 1, one has the following identity:

$$\mathrm{H}^k\,(\mathfrak{g}_e, \mathrm{Hom}\,(W, \mathfrak{g})) = \mathrm{H}^k\,(\mathfrak{g}_e, \mathrm{Hom}\,(W, \Delta_e\mathfrak{g})) \oplus \mathrm{H}^k\,(\mathfrak{g}_e, \mathrm{Hom}\,(W, \tilde{\mathfrak{g}}_e))$$

This digression might be combined with the Hochschild-Serre spectral sequence to be useful in calculating the cohomology spaces $\mathrm{H}^*\,(\mathfrak{g}, \mathrm{Hom}\,(W, \mathfrak{g}))$. However, a direct calculation of $\mathrm{H}^*\,(\mathfrak{g}, \mathrm{Hom}\,(W, \mathfrak{g}))$ is the purpose of the next sections.

3. Cohomology of polynomial maps

Let $\mathcal{S}(\mathfrak{g}))$ be the symmetric algebra of the vector space $\mathfrak{g}$. Our purpose is to calculate the cohomology space $\mathrm{H}^2(\mathfrak{g}, \mathrm{Hom}(W, \mathfrak{g}))$ where W is a homogeneous submodule of the graded $\mathfrak{g}$-module:

$$\mathcal{S}^+(\mathfrak{g}) = \bigoplus_{i\geq 0} \mathcal{S}^i(\mathfrak{g})$$

In the sequel, se shall identify $\mathrm{Hom}(W, f)$ with $W^* \otimes F$, where W^* is the dual vector space of W and $F \in \{\Delta_e\mathfrak{g}, \mathfrak{g}, \tilde{\mathfrak{g}}_e\}$ According to the exact sequence 2, we have the classical long exact sequence:

(3)
$$\begin{array}{ccccc} \mathrm{H}^i(\mathfrak{g}, W^*\otimes\Delta_e\mathfrak{g}) & \longrightarrow & \mathrm{H}^i(\mathfrak{g}, W^*\otimes\mathfrak{g}) & \longrightarrow & \mathrm{H}^i(\mathfrak{g}, W^*\otimes\tilde{\mathfrak{g}}_e) \\ & \swarrow & & & \\ \mathrm{H}^{i+1}(\mathfrak{g}, W^*\otimes\Delta_e\mathfrak{g}) & \longrightarrow & \dots & & \end{array}$$

Before pursuing, let us give some examples of affinelike Lie algebras.

Example 1. Let V be a vector space over $\mathbb{K}$, then the Lie algebra:

$$\mathrm{Aff}(V) = \mathfrak{gl}(V) \ltimes V$$

is an ALLA where e is the identity endomorphism of V.

Example 2. Let $(V, \langle ., .\rangle)$ be a real euclidean space. Its Lie algebra of affine endomorphisms with conformal linear part is an ALLA.

Example 3. Let $\mathfrak{g} = \mathbb{K}\mathrm{Id}_{\mathbb{K}^m} \ltimes K^m$. Then $\mathfrak{g}$ is an ALLA.

Example 4. Let $M(m, n)$ be the vector space of $m \times n$ matrices with entries in $\mathbb{K}$. Let:

$$\mathfrak{g}_{m,n} = \mathfrak{gl}(\mathbb{K}^m) \ltimes M(m, n)$$

with bracket:

$$[(a, b), (a', b')] = ([a, a'], ab' - a'b)$$

Let e be the unit matrix in $\mathfrak{gl}(\mathbb{K}^m)$. Then $(\mathfrak{g}, e)$ is an ALLA.The algebras $\mathfrak{g}_{m,n}$ are sometimes called affine algebras. Their coadjoint representations are deeply studied in [19].

Remark 2. Let $\mathfrak{g}$ be an ALLA. Then:

$$\mathrm{H}^2(\mathfrak{g}, \mathbb{K}) = \mathrm{H}^2([\mathfrak{g}_e, \mathfrak{g}_e], \mathbb{K})$$

In fact, let $e \in \mathfrak{g}$ be an element satisfying properties (alla 1) and (alla 2). Given a scalar 2-cocyle $\theta \in C^2(\mathfrak{g}, \mathbb{K})$, the cartan formula:

$$L(e) = i(e)\partial + \partial i(e)$$

implies that:

$$L(e)\theta = \partial i(e)\theta.$$

Now, let $x = s + u, x' = s' + u' \in \mathfrak{g}_e \oplus \Delta_e\mathfrak{g}$. One easily verifies that:

$$\theta(u, u') = \theta(e, [s, s']) = 0$$

The restriction map from $\mathrm{H}^2(\mathfrak{g}, \mathbb{K})$ to $\mathrm{H}^2(\mathfrak{g}_e, \mathbb{K})$ is thus one-to-one. To see that it is onto, we remark that (alla 2) implies that:

$$\mathfrak{g}_e = \mathbb{K}e \oplus [\mathfrak{g}_e, \mathfrak{g}_e]$$

Moreover, the same (alla 2) shows that the Lie algebra $[\mathfrak{g}_e, \mathfrak{g}_e]$ is perfect. Thus, we have:

$$\mathrm{H}^1(\mathfrak{g}_e, \mathbb{K}) = 0$$

and

$$\left(\mathrm{H}^1(\mathbb{K}e, \mathbb{K})\right)^{\mathfrak{g}_e} = 0$$

Owing to the Hochschild-Serres exact sequence [14], one sees that the inflation map from $\mathrm{H}^2([\mathfrak{g}_e, \mathfrak{g}_e], \mathbb{K})$ to $\mathrm{H}^2(\mathfrak{g}_e, \mathbb{K})$ is injective, thus proving the claim.

In the particular case where $\mathfrak{g} = \mathrm{Aff}(m)$ and $e = Id_{\mathbb{K}^m}$, we have $g_e = \mathfrak{gl}(m, \mathbb{K})$ and:

$$\mathrm{H}^2(\mathfrak{gl}(m, \mathbb{K}), \mathbb{K}) = \sum_{p+q=2} \mathrm{H}^p(\mathbb{K}e, \mathbb{K}) \otimes \mathrm{H}^q(\mathfrak{sl}(m, \mathbb{K}), \mathbb{K}) = 0$$

The reader may consult [MED-REV] for another proof of the latter vanishing theorem, namely:

$$\mathrm{H}^2(\mathrm{Aff}(\mathbb{K}^m), \mathbb{K}) = 0$$

The proof given here is simpler. Another remark to be made is that every affine group:

$$\mathrm{Aff}(\mathbb{K}^m) = \mathrm{GL}(\mathbb{K}^m) \ltimes \mathbb{K}^m$$

has open coadjoint orbits.In other words, every affine group $\mathrm{Aff}(V)$ carries a left invariant symplectic structure, [19]. Indeed, it is sufficient to identify the dual vector space of the Lie algebra $\mathfrak{aff}(m)$ with itself under the transposition map. In other words, considering $\mathfrak{aff}(m)$ as a subspace of $\mathfrak{gl}(m+1, \mathbb{K})$, one deals with the inner product:

$$\langle A, B \rangle = \mathrm{Tr}(A^t B)$$

Let:

$$(A, v) \in \mathfrak{gl}(m) \ltimes \mathbb{K}$$

be a m-cyclic couple in the following sense: $\{v, Av, \ldots, A^{m-1}v\}$ is a basis of the vector space K^m. The couple (A, v) being regarded as an element of $\mathfrak{gl}(m+1, \mathbb{K})$, its transpose, say $(A, v)^t$, is an element of the dual space of $\mathfrak{aff}(m)$. The property to be m-cyclic implies that the coadjoint orbit of $(A, v)^t$ is an open subset of the dual vector space of $\mathfrak{aff}(m)$. For instance, take the square $m \times m$ matrix $A = (a_i j)$ whose nonzero entries are:

$$\begin{cases} a_{i+1,i} = 1 \\ a_{1,m} = 1 \end{cases}$$

Thus, A is an m-cyclic matrix. Take the vector $v = (1, 0, .., 0) \in \mathbb{K}^m$. Then, the couple (v, A) being regarded as an element of the dual space of $\mathfrak{aff}(m)$ has an open coadjoint orbit.

4. Weak nondegeneracy

Let Π be a smooth Poisson tensor on the smooth manifold $\mathbb{K}^m$. Let us suppose 0 to be a singularity of Π and let $\hat{\Pi}$ be the Taylor expansion at 0 of Π. The linear part of Π, say Π^1, defines a Lie algebra structure in the vector space of linear functions. Let us denote this Lie algebra structure by $(\mathbb{K}^{*m}, \Pi^1)$. Following Weinstein, a Lie algebra $\mathfrak{g}$ is called formally (resp. analytically or smoothly) non degenerate if every formal (resp. analytic or smooth) Poisson tensor Π satisfying both conditions:

$$\Pi(0) = 0$$

and

$$(\mathbb{K}^{*m}, \Pi^1) = \mathfrak{g}$$

is formally (resp. analytically or smoothly) linearizable at 0. In other words, Π is formally (resp. analytically or smoothly) linearizable at 0 if (near 0) there exist a system of formally (resp. analytic or smooth) coordinate functions:

$$F := (f_1, \ldots, j_m)$$

such that:

$$\Pi\left(df_i, df_j\right) = \sum_k c_{ij}^k f_k$$

with $c_{ij}^k \in \mathbb{K}$. Now, to every smooth Poisson tensor Π on a smooth manifold M is assigned the Lie algebroid structure $(T^*M, \natural)$. The anchor map $\natural$ is the vector bundle homomorphism from T^*M to TM defined by:

$$\natural(\alpha) = i(\alpha)\Pi.$$

The anchor map $\natural$ and the Poisson tensor Π have the same singularities. In particular $\natural(0) = 0$. The linear part of the Taylor expansion at 0 of $\natural$ defines a Lie algebra structure in the cotangent space T_0^*M. That Lie algebra structure is the same as the one defined by Π^1. So, it is easy to see that the Taylor expansion at 0 of $\natural$ defines a formal deformation of the linear Lie algebroid:

$$\mathfrak{g} \times T_0 M$$

where:

$$\mathfrak{g} = \left(T_0^* M, \Pi^1\right).$$

Definition 2. A Lie algebra $\mathfrak{g}$ is called weakly formally (resp. weakly analytically or weakly smoothly) non degenerate in the sense of Weinstein if the following conditions hold: let Π be a formal (resp. analytical or smooth) Poisson structure such that:

$$\begin{cases} \Pi(0) = 0. \\ \mathfrak{g} = \left(T_0^* M, \Pi^1\right). \end{cases}$$

then the corresponding formal (resp. analytic or smooth) Lie algebroid structure $(T^*M, \natural)$ is formally (resp.analytically or smoothly) linearizable at 0.

The formal (resp. analytic or smooth) linearization at 0 of Π implies the formal (resp. analytic or smooth) linearization at 0 of its associated Lie algebroid. The converse doesn't hold, even formally.

5. Main results

From now on, we shall represent an ALLA by the couple $(\mathfrak{g}, e)$. We fix once for all an homogeneous submodule W of the graded $\mathfrak{g}$-module $S^{+}(\mathfrak{g})$. Then, one has the following exact sequence of $\mathfrak{g}$-modules:

$$0 \longmapsto \mathrm{Hom}\,(W, \Delta_e\mathfrak{g}) \longmapsto \mathrm{Hom}\,(W, \mathfrak{g}) \longmapsto \mathrm{Hom}\,(W, \tilde{\mathfrak{g}}_e) \longmapsto 0 \tag{4}$$

We shall denote by W^l the subspace of W consisting of homogeneous elements of degree l. Let $Z\,(\Delta_e\mathfrak{g})$ be the centralizer of $\Delta_e\mathfrak{g}$ in the Lie algebra $\mathfrak{g}$ and let $R\,(\Delta_e\mathfrak{g})$ be the radical of the Lie algebra $\Delta_e\mathfrak{g}$. Keeping these notations, the main concern of this section is to prove the next statements:

Theorem 3. *Let $(\mathfrak{g}, e)$ be a non solvable affinelike Lie algebra. Then, for every positive integer l, the following Chevalley-Eilenberg cohomology spaces vanish:*

$$H^2\left(\mathfrak{g}, Hom\left(W^l, \Delta_e\mathfrak{g}\right)\right).$$
$$H^2\left(\mathfrak{g}, Hom\left(W^l, \tilde{\mathfrak{g}}_e\right)\right).$$

Theorem 4. *Let $(\mathfrak{g}, e)$ be a non solvable affinelike Lie algebra.Then, for every homogeneous submodule V of the graded $\mathfrak{g}$-module $S^{+}\mathfrak{g}$:*

$$Ext^2_{\mathfrak{g}}\,(V, \mathfrak{g}) = 0$$

Theorem 5. *Every non solvable affinelike Lie algebra is formally (resp. analytically) weakly non degenerate in the sense of Alan Weinstein.*

Before stating the proofs of the theorems which are stated above, it is useful to fix some notations. Let $\mathfrak{g}_e = R\,(\mathfrak{g}_e) + S$ be a fixed Levi decompsition of $\mathfrak{g}_e$. Therefore, the vector space $\mathfrak{g}$ may be decomposed as folows:

$$\mathfrak{g} = \Delta_e\mathfrak{g} + R\,(\mathfrak{g}_e) + S$$

Each subspace W^l is multi-graded as indicated below:

$$W^l = \bigoplus W^{t,r,s}$$

where t, r and s are non negative integers such that $t + r + s = l$ and $W^{t,r,s}$ is a subspace of the vector space:

$$S^t\,(\Delta_e\mathfrak{g}) \otimes S^r\,\left(R\,(\mathfrak{g}_e)\right) \otimes S^s\,(S)\,.$$

Every vector space $C^k\left(\mathfrak{g}, \mathrm{Hom}\left(W^l, \Delta_e\mathfrak{g}\right)\right)$ is a $\mathfrak{g}$-module under the prolongation of the adjoint representation. Thus, for each $a \in \mathfrak{g}$, the action of a (resp. the inner product by a) is denoted by $L_{(a)}$ (resp. $i(a)$). The Chevalley-Eilenberg coboundary operator, namely ∂, is related to $L_{(a)}$ by the next Cartan formula:

$$L_{(a)} = \partial i(a) + i(a)\partial$$

Let $\xi \in W^l$. According to the previous notations, we shall write ξ as follows:

$$\xi = \sum_{t,r,s} \xi^{t,r,s}$$

It is easy to see that $L_{(e)}\xi^{t,r,s} = t\xi^{t,r,s}$. Now, let $\theta \in Z^2\left(\mathfrak{g}, W^l \otimes \Delta_e\mathfrak{g}\right)$. Following the Cartan formula we get:

$$L_{(e)}\theta = \partial i(e)\theta.$$

Before pursuing, let $\bigwedge^k \mathfrak{g}$ be the k-th exterior power of the vector space $\mathfrak{g}$. Then, we can decompose it as follows:

$$(\spadesuit): \bigwedge^k \mathfrak{g} = \bigoplus \bigwedge^m \Delta_e\mathfrak{g} \wedge \bigwedge^n R(\mathfrak{g}_e) \wedge \bigwedge^p S$$

with $m + n + p = k$. We are now in position to prove theorem 3.

Part 1

Step 1

Proof. Let $u, v \in \Delta_e\mathfrak{g}$ and $\xi^{t,r,s} \in W^{t,r,s}$. A direct calculation of:

$$\left(\left(L_{(e)}\theta\right)(u, v)\right)\left(\xi^{t,r,s,}\right)$$

yields the next result:

$$\left(\left(L_{(e)}\theta\right)(u, v)\right)\left(\xi^{t,r,s,}\right) = (-1 + t)\left(\theta(u, v)\right)\left(\xi^{t,r,s}\right)$$

For each $\phi \in W^l \oplus \Delta_e\mathfrak{g}$, the $W^{t,r,s}$ component of ϕ is denoted by $\phi^{t,r,s}$. The calculations made above show that for every 2-cocycle:

$$\theta \in C^2\left(\mathfrak{g}, W^l \oplus \Delta_e\mathfrak{g}\right)$$

one has:

$$(-1 + t)\,\theta^{t,r,s}(u, v) = (\partial i(e)\theta)(u, v).$$

From the identity above we deduce that if $t \neq 1$, then the map $(u, v) \mapsto \theta^{t,r,s}(u, v)$ coincides with the boundary of a 1-cochain.

Step 2

Let $x, x' \in \mathfrak{g}$. One has the following identity:

$$\left(\left(L_{(e)}\theta\right)(x, x')\right)\left(\xi^{t,r,s}\right) = (1 + t)\left(\theta(x, x')\right)\left(\xi^{t,r,s}\right)$$

Since t is a non negative integer, the map $(x, x') \mapsto \theta(x, x')$ coincides with the coboundary of a 1-cochain.

Step 3

It remains to examine $\theta(x, u)$ for $(x, u) \in \mathfrak{g}_e \oplus \Delta_e\mathfrak{g}$. Let us fix $\xi^{t,r,s}$ as above. It comes:

$$\left(\left(L_{(e)}\theta\right)(x, u)\right)\left(\xi^{t,r,s}\right) = t\left(\theta(x, u)\right)\left(\xi^{t,r,s}\right)$$

Using arguments similar to those used in *step 2*, we deduce that if t is a positive integer, then there is a 1-cochain $\phi(t) \in C^1\left(\mathfrak{g}, \mathrm{Hom}\left(W^l, \Delta_e\mathfrak{g}\right)\right)$ such that:

$$\theta(x, u) = \partial\phi(t)(x, u).$$

Step 4

Now, we take into account the decompsition $(\spadesuit)$. Given a cochain $\theta \in C^{m+n+p}(\mathfrak{g}, W \otimes \mathfrak{g})$, its restriction to $\bigwedge^{m,n,p} \mathfrak{g}$ is denoted by $\theta_{m,n,p}$. Therefore, every k-cochain:

$$\theta^k \in C^k\left(\mathfrak{g}, \mathrm{Hom}\left(W^l, \Delta_e\mathfrak{g}\right)\right)$$

is decomposed as follows:

$$\theta^k = \sum_{m+n+p=k} \sum_{t+r+s=l} \theta^{t,r,s}_{m,n,p}$$

In particular, when ϕ is a 2-cocyle, *step 1, step 2* and *step 3* tell us that all its components $\theta^{t,r,s}_{m,n,p}$ are exact, except when $t = 0$ or $t = 1$. So, every cohomology class $\left[\theta^l\right] \in \mathrm{H}^2\left(\mathfrak{g}, \mathrm{Hom}\left(W^l, \Delta_e\mathfrak{g}\right)\right)$ may be represented by a cocycle of the form:

$$\sum_{r+s=l} \theta^{1,r,s}_{2,0,0} + \sum_{n+p=1} \sum_{r+s=l-1} \theta^{0,r,s}_{1,n,p}$$

New, let's consider the map $\chi \colon S \to \mathrm{Hom}\left(\Delta_e\mathfrak{g} + R\left(\mathfrak{g}_e\right), S^{0,r,s}(\mathfrak{g}) \otimes \Delta_e\mathfrak{g}\right)$ that maps x to $\sum_{m+n=1}\sum_{r+s=t-1} i(x)\theta^{0,r,s}_{m,n,1}$. By virtue of Cartan formula, this map is a cocycle of S. Since S is semi-simple, there exists an element $\phi \in \mathrm{Hom}\left(\Delta_e\mathfrak{g} + R(\mathfrak{g}_e), S^l(\mathfrak{g}) \otimes \Delta_e\mathfrak{g}\right)$ whose coboundary is χ. Now, we are in position to prove that the cohomology $\left[\theta^k\right]$ may be represented by a cocycle of the form:

$$\sum_{r+s=l-1} \theta^{1,r,s}_{2,0,0} + \sum_{r+s=l} \theta^{0,r,s}_{1,1,0}.$$

Indeed, let us consider the cochain $\mu \in C^1\left(\mathfrak{g}, S^l(\mathfrak{g}) \otimes \Delta_e\mathfrak{g}\right)$ which is defined by:

$$\mu(x, y) = \phi(y).$$

where $x \in S$ and $y \in \Delta_e\mathfrak{g} + R\left(\mathfrak{g}_e\right)$. A direct calculation shows that the component:

$$\sum_{m+n=l} \sum_{r+s=l} \theta^{0,r,s}_{m,n,1}$$

is the coboundary of μ.Therefore, the cohomology class of $\left[\theta^l\right]$ may be represented by a cocycle of the form:

$$(\star) = \sum_{r+s=l-1} \theta^{1,r,s}_{2,0,0} + \sum_{r+s=l} \theta^{0,r,s}_{1,1,0}$$

For convenience and without loss of generality, we may assume that the ground field $\mathbb{K}$ is the field of complex numbers $\mathbb{C}$. Fix a Cartan subalgebra $\mathfrak{h} \subset S$.To the couple $(\mathfrak{h}, S)$ we assign the roots system $\mathcal{R}$ (resp. $\mathcal{R}^{p}\prime)$) of the S-module $\Delta_e\mathfrak{g} + R\left(\mathfrak{g}_e\right)$ (resp. $S^l\mathfrak{g} \otimes \Delta_e\mathfrak{g}$). The elements of $\mathcal{R}$ (resp.$\mathcal{R}'$) are labelled α_j (resp. γ_j). t is to be noticed that the multigraduation that we are using is preserved by the action of $\mathfrak{g}_e$. Thus, given a cochain θ and a root $\gamma \in \mathcal{R}'$, the γ-component of θ is denoted by $\theta^{(\gamma)}$). Now, let $\alpha_i, \alpha_j \in \mathcal{R}$, and $\gamma \in \mathcal{R}'$. Let $H \in \mathfrak{h}$. Tanking into account those notations and the Cartan formula, one easily deduces from the expression $(\star)$ the following identities;

$$\gamma\left(H\right)\theta^{(\gamma)}\left(x_{\alpha_i}, x_{\alpha_j}\right) = \left(\alpha_i + \alpha_j\right)(H)\theta^{(\gamma)}\left(\alpha_i, \alpha_j\right).$$

Those identities hold if and only if $\theta^{(\gamma)} = 0$, $\forall \gamma \in \mathcal{R}'$. That ends the proof of the vanishing of $\mathrm{H}^2\left(\mathfrak{g}, W \otimes \Delta_e\mathfrak{g}\right)$.

Part 2

The concern of Part2 is to prove that $\forall l > 1$, the cohomology space $\mathrm{H}^2\left(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e\right)$ vanishes. The proof is similar to that in Part 1 and so we shall keep the same notations.

Step 1

Let θ be a 2-cocycle in $C^2\left(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e\right)$. Given $u, v \in \Delta_e\mathfrak{g}$, using the Cartan formula, a direct calculation of $\left(\left(L_{(e)}\theta^{t,r,s}\right)\right)(u, v)$ yields:

$$(t-2)\theta^{t,r,s}(u, v) = \left(\partial i(e)\theta\right)(u, v).$$

Therefore, we deduce that the quantity $\theta(u, v)$ coincides with the coboundary of a 1-cochain when $t \neq 2$. So, except $\theta^{2,r,s}_{2,0,0}$, all of the components $\theta^{t,r,s}_{2,0,0}$ are exact.

Step 2.

Given $x, x' \in \tilde{\mathfrak{g}}_e$, one deduces from the identity:

$$\left(\left(L_{(e)}\theta\right)(x,x')\right) = \left(\partial i(e)\theta\right)\theta(x,x')$$

the following relation:

$$t\theta^{t,r,s}(x,x') = \left(\partial i(e)\theta\right)(x,x').$$

So, if $t \neq 0$, the component $\theta^{t,r,s}_{0,n,p}$ coincides with the component of an exact cocycle.
Step 3

Given $(x,u) \in \tilde{\mathfrak{g}}_e \oplus \Delta_e\mathfrak{g}$, we have:

$$(t-1)\theta^{t,r,s}(x,u) = \left(\partial i(e)\theta\right)(x,u).$$

Thus, except $\theta^{1,r,s}_{1,n,p}$ all of the components $\theta^{t,r,s}_{1,n,p}$ coincides with the restriction of exact cocycle. Now, we can conclude from the three steps above that every cohomology class

$$[\theta] \in \mathrm{H}^2\left(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e\right)$$

may be represented by a cocycle of the following form:

$$\theta = \sum_{r+s=l-2}\theta^{2,r,s}_{2,0,0} + \sum_{n+p=1}\sum_{r+s=l-1}\theta^{1,r,s}_{1,n,p} + \sum_{r+s=l}\sum_{n+p=2}\theta^{0,r,s}_{0,n,p}.$$

Now, taking into account this reduced form of θ, it is easy to verify that the following component of θ:

$$\phi = \sum_{r+s=l}\theta^{0,r,s}_{0,0,2}$$

is a cocycle of S. Since S is semi-simple, there exists an element $\chi \in C^1\left(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e\right)$ whose boundary is ϕ. Thus, we can represent the cohomology class of θ^l by:

$$(\star\star) = \sum_{r+s=l-2}\theta^{2,r,s}_{2,0,0} + \sum_{r+s=l-1}\left(\theta^{1,r,s}_{1,1,0} + \theta^{1,r,s}_{1,0,1}\right) + \sum_{r+s=l}\left(\theta^{0,r,s}_{0,2,0} + \theta^{0,r,s}_{0,1,1}\right).$$

Now, we consider the map μ (resp. ν) from S to $\mathrm{Hom}\left(\Delta_e\mathfrak{g}, W \otimes \tilde{\mathfrak{g}}_e\right)$ (resp. from S to $\mathrm{Hom}\left(R\mathfrak{g}_e), W \otimes \tilde{\mathfrak{g}}_e\right))$ which is defined as follows:

$$\mu(x) = \sum_{r+s=l-1} i(x)\theta^{1,r,s}_{1,0,1}$$

and:

$$\nu(x) = \sum_{r+s=l} i(x)\theta^{0,r,s}_{0,1,1}.$$

Both μ and ν are cocycles of the semi-simple Lie algebra S. Thus, it exists $\xi \in \mathrm{Hom}\left(\Delta_e\mathfrak{g}, W \otimes \tilde{\mathfrak{g}}_e\right)$ (resp. $\zeta \in \mathrm{Hom}\left(R(\mathfrak{g}_e), W \otimes \tilde{\mathfrak{g}}_e\right)$) whose coboundary is μ (resp. ν). Now, let one regard the map $\eta = \xi + \zeta$ as an element of $C^1(\mathfrak{g}, W \otimes \mathfrak{g}_e)$ by setting:

$$\eta(u+y+x) = \xi(x) + \zeta(x),\ \forall (u,y,x) \in \Delta_e\mathfrak{g} \oplus R(\mathfrak{g}_e) \oplus S.$$

Now, the cocycle $\theta' = (\star\star) - d\eta$ doesn't contain any component of type $\theta^{t,r,s}_{0,0,p}$ with $p > 0$. Thus, any cohomology class in $\mathrm{H}^2\left(\mathfrak{g}, W^l \otimes \tilde{\mathfrak{g}}_e\right)$ may be represented by a cocycle of the following REDUCED form:

$$\theta' = \sum_{r+s=l-1}\theta^{1,r,s}_{2,0,0} + \sum_{r+s=l}\left(\theta^{0,r,s}_{1,1,0} + \theta^{0,r,s}_{0,2,0}.\right)$$

Step 4.

We assume again $\mathbb{K} = \mathbb{C}$. Therefore, we fix a Cartan subalgebra $\mathfrak{h} \subset S$. We label α_i (resp. γ_j) the corresponding roots system of the S-module $\Delta_e\mathfrak{g} + R(\mathfrak{g}_e)$ (resp.$\sum_{r+s=l-1} W^{1,r,s} \otimes$

$\tilde{\mathfrak{g}}_e + \sum_{r+s=l} W^{0,r,s} \otimes \tilde{\mathfrak{g}}_e$). Now, let α_i, $\alpha_{i'}$ and γ be roots. Taking into account the REDUCED form θ' of 2-cocycles, we denote by $\theta'_{(\gamma)}$ the γ-component of θ'. Then the Cartan formula implies the following identities:

$$\gamma(H)\theta'_{(\gamma)}\left(u_{\alpha_i}, u_{\alpha_{i'}}\right) = (\alpha_i + \alpha_{i'})(H)\theta'_{(\gamma)}\left(u_{\alpha_i}, u_{\alpha_{i'}}\right), \ \forall \alpha_i, \alpha_{i'}, \gamma.$$

Of course, those identities hold if and only if each $\theta'_{(\gamma)}$ vanishes identically. □

Let us now proceed to the proof of theorem 4.

Proof. The notations used will be the ones previously defined.Fix an homogeneous submodule $W \subseteq S^+\mathfrak{g}$ and let k be a non negative integer. Owning to Remark 1, we have:

$$\mathrm{Ext}^k_{\mathfrak{g}}(W, F) = \bigoplus_l \mathrm{Ext}^k_{\mathfrak{g}}\left(W^l, F\right)$$

where l runs over the set of positive integers and $F \in \{\Delta_e\mathfrak{g}, \mathfrak{g}, \tilde{\mathfrak{g}}_e\}$. On one hand, (2) gives rise the classical long cohomology exact sequence. In particular, the following sequence is exact at the level $\mathrm{H}^2\left(\mathfrak{g}, \mathrm{Hom}\left(W^l, \mathfrak{g}\right)\right)$:

$$\mathrm{H}^2\left(\mathfrak{g}, \mathrm{Hom}\left(W, \Delta_e\mathfrak{g}\right)\right) \longmapsto \mathrm{H}^2\left(\mathfrak{g}, \mathrm{Hom}\left(W, \mathfrak{g}\right)\right) \longmapsto \mathrm{H}^2\left(\mathfrak{g}, \mathrm{Hom}\left(W, \tilde{g}_e\right)\right)$$

On the other hand, if F is a finite dimensional $\mathfrak{g}$-module, then we have the following classical linear isomorphisms:

$$\mathrm{Ext}^k_{\mathfrak{g}}\left(W^l, F\right) \sim \mathrm{H}^k\left(\mathfrak{g}, \mathrm{Hom}\left(W^l, F\right)\right)$$

Thus, owning to Theorem (3), the exact sequence above yields

$$\mathrm{Ext}^2_{\mathfrak{g}}(W, \mathfrak{g})$$

□

Before proceeding, let us recall the meaning of Theorem (5). Let Π be a smooth Poisson tensor defined in a m-dimensional smooth manifold M. Theorem (5) expresses a local property of Lie algebroids given by Poisson structures (near to their singularities). Thus, we shall suppose that M is an open neighborhood of the origin of $\mathbb{K}^m$. Let us suppose thus that Π vanishes at the origin 0 of $\mathbb{K}^m$. Let:

$$\check{\Pi} = \sum_I \Pi^I$$

be the Taylor expansion at 0 of Π, where all of the I are multi-indices, namely $I = (i_1, \ldots, i_m)$. Let Π^1 be the linear part of Π. Then, Π^1 defines a Lie algebra structure in the vector space of linear functions. The Taylor expansion at 0 of Π is regarded as a formal deformation of Π^1. In fact, we have three Poisson structures in a small neighborhood of 0, namely, the smooth structure Π, the formal structure $\check{\Pi}$ and the linear structure Π^1. So arise the questions to know whether Π (resp. $\check{\Pi}$)is smoothly (resp. formally) isomorphic to Π^1.

Proposition 3. *If Π is smoothly isomorphic to Π^1, then $\check{\Pi}$ will be formally isomorphic to Π^1.*

Proof. Given a multi-index $I = \left(i_1, \ldots, i_p\right)$, we shall set:

$$l(I) = \sum_{j=1}^{p} i_j$$

Let us denote by $\mathfrak{g}$ the Lie algebra defined by Π^1. Then, each homogeneous component Π^I of the Taylor expansion at 0 of Π is an element of $C^2\left(\mathfrak{g}, S^{l(I)\mathfrak{g}}\right)$. Let us consider the Lie algebroid structure:

$$A_\pi = (T^*M, \natural)$$

which is defined by (M, Π). Let us recall the following facts. Firstly, the anchor map $\natural$ is the vector bundle map defined by:

$$\natural(\alpha) = i(\alpha)\Pi.$$

Secondly, given 2 smooth sections of T^*M, say α, β their bracket is defined by:

$$[\alpha, \beta]_\natural = i(\natural\alpha)d\beta - i(\natural\beta)d\alpha + di(\natural\alpha)\beta.$$

Thus, one easily sees that a Poisson tensor and the associated Lie algebroid have the same singularities and the same linear part at their singular points. Let $\check{A}_\pi$, be the Taylor expansion at 0 of the anchor map $\natural$. It may be regarded as a formal deformation of the linear Lie algebroid:

$$\mathfrak{g} = \left(T_0^*M, \Pi^1\right) \times T_0M$$

Now, $\tilde{x}$ let be a smooth vector field defined near 0, let $\tilde{\alpha}$ be a smooth section of T^*M. Let us set:

$$\alpha = \tilde{\alpha}(0)$$

and:

$$x = \tilde{x}(0)$$

Then, the expression:

$$\rho(\alpha)x = [\natural(\tilde{\alpha}), \tilde{x}]\,(0)$$

is well defined. It is easy to verify that:

$$\rho\left([\alpha, \beta]_\natural\right) = [\rho(\alpha), \rho(\beta)]\,.$$

Therefore, the dual vector space:

$$\mathfrak{g}^* = T_0M$$

is a $\mathfrak{g}$-module, but the action of $\mathfrak{g}$ is not the coadjoint action. The (graded) $\mathfrak{g}$-module structure of $S^+\mathfrak{g} \otimes \mathfrak{g}^*$ is inherited from the tensor product $\mathrm{ad}_\mathfrak{g} \otimes \rho$. Moreover, the coefficients of the Taylor expansion at 0 of $\natural$ may be regarded as elements of the Spencer prolongations of the linear subalgebra $\rho(\mathfrak{g}) \subset \mathfrak{gl}(T_0) \subset \ldots$, [13]. From this viewpoint, the algebroid $(T^*M, \natural)$ is formally linearizable at 0 if the first prolongation of $\rho(\mathfrak{g})$ is zero. So it is the case when $\rho(\mathfrak{g})$ is compact or when the fundamental form

$$B_\rho(\alpha, \beta) = \mathrm{Tr}\left(\rho(\alpha)\rho(\beta)\right)$$

is non degenerate, [13]. What is just said above is nothing but the algebraic deformation theoretic point of view applied to the linearization problem for the anchor map $\natural$. We can also examine the linearization of $(T^*M, \natural)$ from the abstract algebra viewpoint. Indeed, let us denote by $\Omega(M)$ the vector space of smooth sections of T^*M. For every non negative integer 1, we set:

$$\omega^{(l)} = \{\alpha \in \Omega(M) | j_0^l\alpha = 0\}$$

Every $\Omega^{[l)}$ is an ideal of the Lie algebra structure associated to the Lie algebroid $(T^*M, \natural)$. Therefore, we get a natural filtration:

$$\Omega(M) \supset \Omega^{(1)}(M) \supset \Omega^{(2)}(M) \supset \dots$$

From now on, we shall keep in mind this filtered Lie algebra structure in $\Omega(M)$, [15]. It is easily seen that the quotient space $\Omega^{(l)}/\Omega^{(l+1)}$ may be canonically identified with:

$$S^l T_0^* M \otimes T_0^* M = \operatorname{Hom}\left(S^l T_0 M, T_0^* M\right).$$

Now, let us denote by grd Ω the graded Lie algebra whose l-th homogeneous subspace is:

$$\operatorname{grd}{}^l\Omega = \Omega^{(l)}/\Omega^{(l+1)}.$$

In fact, the positive part of grd Ω, namely:

$$\operatorname{grd}{}^+\Omega = \bigoplus_{l\geq 1} \Omega^{[l)]}/\Omega^{(l+1)}$$

is an ideal of grd Ω. Moreover, the quotient Lie algebra:

$$\operatorname{grd}\Omega/\operatorname{grd}{}^+\omega$$

is nothing but the Lie algebra:

$$\left(T_0^* M, [.,.]_\natural\right) = \left(T_0^* M, \Pi^1\right).$$

So, we have the following exact sequence of Lie algebras

$$(\heartsuit)\colon 0 \to \operatorname{grd}{}^+\Omega \to \operatorname{grd}\Omega \to \left(T_0^* M, [.,.]_\natural\right) \to 0$$

The problem to know whether the sequence above (Lie) splits is the formal linearization problem for the Poisson structure (M, Π), see [21]. The study of this problem involves the cohomology space $\mathrm{H}^2(\mathfrak{g}, \operatorname{grd}^+\Omega)$, [18, 20]. So, by the virtue of Theorem (4), the exact sequence of Lie algebras $(\heartsuit)$ splits. In other words, there exists a Lie algebra homomorphism ϱ from $\left(T_0^* M, [.,.]_\natural\right)$ to the formal Poisson algebra $\left(j_0^\infty\left(C^\infty(M,\mathbb{R})\right), [.,.]\right)$ such that $d_0\varrho(\lambda) = \lambda$, $\forall \lambda \in T_0^* M$. Of course, the bracket on power series is induced by the Poisson structure (M, Π), viz:

$$\left[j_0^\infty f, j_0^\infty g\right] = j_0^\infty\left(\Pi(f, g)\right)$$

Since $\Pi(f, h)$ depends only on the differential of the both f and h, we may restrict our attention to the set $C_0^\infty(M,\mathbb{R})$ of real valued smooth or analytic functions which vanish at the point $0 \in M$. The differentiation at $0 \in M$, say d_0, induces a Lie algebra homomorphism from $C_0^\infty(M,\mathbb{R})$ onto $\left(T_0^* M, [.,.]_\natural\right)$. So, the smooth or analytic linearization problem for Π is to know whether the following exact sequence of Lie algebra splits:

$$0 \longrightarrow \ker(d_0) \longrightarrow C_0^\infty(M,\mathbb{R}) \longrightarrow \left(T_0^* M, [.,.]_\natural\right) \longrightarrow 0$$

On the other side, from the deformation theory viewpoint, a sufficient condition for the formal linearization of $(T^*M, \natural)$ (resp. Π) is the vanishing of the cohomology space $\mathrm{H}^2(\mathfrak{g}, \operatorname{Hom}(W, \mathfrak{g}))$ (resp. $\mathrm{H}^2(\mathfrak{g}, W^*)$) for every homogeneous submodule W of the graded $\mathfrak{g}$-module $S^+(\mathfrak{g}^*)$. Now, let us suppose the Lie algebra:

$$\mathfrak{g} = \left(T_0^* M, \Pi^1\right)$$

to have an affinelike Lie algebra structure, namely $(\mathfrak{g}, e)$. Let us denote by $\mathfrak{g}_f$ the formal deformation of $\mathfrak{g}$ defined by the Taylor expansion at 0 of the anchor map $\natural$. Its bracket is in a basis $(dx_1, \ldots, dx_m)$ is defined as follows:

$$\left[dx_i, dx_j\right] = \sum_k \left(\sum_I c_{ij}^{I,k}(0) x^I\right) dx_k$$

Where $c_{ij}^{I,k}(0)$ is regarded as an element of Hom $\left(S^{l(I)}\mathfrak{g}^*, \mathfrak{g}\right)$. Keeping the notations of the previous sections, we deduce from Theorem (4) that

$$\mathrm{H}^2\left(\mathfrak{g}, \mathrm{Hom}\,(W, \mathfrak{g})\right) = 0.$$

Thus, the formal Lie algebra structure $\mathfrak{g}_f$ is formally isomorphic to its linear part, namely $\left(T_0^*M, \Pi^1\right)$ □

Remark 3. When the Lie algebra $(T^*M, \natural)$ is an affinelike Lie algebra, it is more relevant to deal with the splitting problem of (♡) rather than to deal with formal diffeomorphisms as Conn (resp. Nguyen Tien Zung) does in the case where $\mathfrak{g}$ is semi-simple. Indeed, when $\mathfrak{g}$ is semi-simple, one has Casimir elements which permit to construct an explicit homotopy operator. That operator yields an explicit formal diffeomorphism linearizing formally the Poisson structure. If $\mathfrak{g}$ is an non solvable affinelike Lie algebra, one proceeds as it follows. For every non negative integer k, let $S_k = S_k^\infty(M, \mathbb{R})$ be the set of formal power series whose orders are at least equal to $k+1$. Every S_k is an ideal of the Lie algebra S_0 and the quotient Lie algebra $\mathfrak{g}_k = S_0/S_k$ may be identified with the vector space $\sum_{1 \le j \le k} S^j(\mathfrak{g})$. In particular one has $\mathfrak{g}_0 = \mathfrak{g}$. Moreover, one has the following exact sequence of Lie algebras:

$$0 \longrightarrow S^{k+1}(\mathfrak{g}) \longrightarrow \mathfrak{g}_k \longrightarrow \mathfrak{g}_{k-1} \longrightarrow 0$$

If $j < k$, the canonical projection of $\mathfrak{g}_k$ onto $\mathfrak{g}_j$ is denoted by p_{jk}. It is easily seen that $p_{ij}p_{jk} = p_{ik}$. Thereby, one has the projective system $(\mathfrak{g}_k, p_{jk})$. In particular the following exact sequence of Lie algebras splits:

$$0 \longrightarrow S^2(\mathfrak{g}) \longrightarrow \mathfrak{g}_1 \longrightarrow \mathfrak{g} \longrightarrow 0$$

Thus, one constructs inductively an Lie algebra monomorphism $\varrho_j : \mathfrak{g} \to \mathfrak{g}_j$ such that for $j < k$, one has $p_{jk}\varrho_k = \varrho_j$. Those considerations yield a Lie algebra monomorphism ϱ from $\mathfrak{g}$ to the inverse limit of $(\mathfrak{g}_k, p_{jk})$, which is $j_0^\infty\left(C_0^\infty(M, \mathbb{R})\right)$. That is nothing but the formal linearization of the Poisson structure, [21].

Now, suppose the Poisson tensor Π to be analytic. Then, the corresponding Lie algebroid is analytic as well. If $\mathfrak{g} = \left(T_0^*M, [.,.]\right)$ is semi-simple or if $\mathfrak{g} = \mathfrak{g}_{n,1}$, then the analytic counterparts of Theorem 5 are nothing but Conn's linearization theorem of analytic Poisson structure, [7, 8, 23] and the recent analytic non degeneracy theorem for $\mathfrak{aff}(n)$ by Jean-Paul Dufour and Nguyen Tien Zung, [10]. If $\mathfrak{g}$ is an affinelike Lie algebra, we know how to construct a Lie algebra monomorphism from $\mathfrak{g}$ to $\mathfrak{g}$. Indeed, let:

$$\sigma : \mathfrak{g} \mapsto \mathfrak{g}_1$$

be a linear monomorphism such that $p_{01}\sigma$ is the identity endomorphism of $\mathfrak{g}$; the bilinear map:

$$\omega(a, b) = \sigma([a, b]) - [\sigma(a), \sigma(b)]\,,\; a, b \in \mathfrak{g}$$

is a $S^2(\mathfrak{g})$-valued 2-cocyle of $\mathfrak{g}$ whose cohomology class doesn't depend on the choice of σ. On one side, the linear map:

$$\bar{\omega}(a) = (a - \sigma(p_{01}(a))) + p_{01}(a)$$

is a Lie algebra isomorphism from $\mathfrak{g}_1$ onto the semi direct product $S^2(\mathfrak{g}) \ltimes \mathfrak{g}$. On the other side, there is an element $\theta \in \mathrm{Hom}\left(\mathfrak{g}, S^2(\mathfrak{g})\right)$ such that:

$$\omega(a,b) = [\sigma(a), \theta(b)] - [\sigma(b), \theta(a)] - \theta\left([a,b]\right).$$

The map ϱ_1 assigning to every $a \in \mathfrak{g}$ the element $\varrho_1(a) = \theta(a) + a$ is an Lie algebra monomorphism from $\mathfrak{g}$ to the semi-direct product $S^2(\mathfrak{g}) \ltimes \mathfrak{g}$. What is just done is nothing but the first step yielding a Lie algebra monomorphism ϱ from $\mathfrak{g}$ to $j^\infty\left(C_0^\infty(M,\mathbb{R})\right)$. Thus, let $x_1, \ldots, x_n$ be a basis of $\mathfrak{g}$; let c_{ij}^k be the structure constants of $\mathfrak{g}$ in the basis $(x_i)_{i=1\ldots n}$, viz:

$$\left[x_i, x_j\right] = \sum_k c_{ij}^k x_k$$

then one has also:

$$\left[\varrho(x_i), \varrho(x_j)\right] = \sum_k c_{ij}^k \varrho(x_k)$$

So, the analytic linearization problem is to prove that one can choose the basic data σ and θ to make sure that the formal series $\varrho(x_1), \ldots, \varrho(x_n)$ are convergent. By the virtue of Conn's theorem (resp. Dufour-Nguen tien Zung's Theorem) such data can be chosen whenever $\mathfrak{g}$ is semi-simple (resp. $\mathfrak{g}$ is the affine Lie algebra $\mathfrak{aff}(n)$. I conjecture that the same conclusion holds for non solvable affinelike Lie algebras.
Proof of theorem 1

Proof. In fact, this is the Poisson counterpart of Theorem 5. Mutatis mutandis, the same arguments used in the proofs of Theorem 5 yield:

$$\mathrm{H}^2\left(\mathfrak{g}, S^+(\mathfrak{g})\right) = 0$$

for every non solvable affinelike Lie algebra $(\mathfrak{g}, e)$. Therefore, such an affinelike Lie algebra is formally nondegenerate. □

Let us now give the proof of Theorem 2.

Proof. We have already proved the equality:

$$\mathrm{H}^2\left(\mathfrak{g}, \mathbb{K}\right) = \mathrm{H}^2\left(\Delta\mathfrak{g}_e, \mathbb{K}\right)$$

We denote the radical of $\Delta_e\mathfrak{g}$ by $R\left(\Delta\mathfrak{g}_e\right)$. Lyndon-Hocschild-Serre spectral sequence yields:

$$\mathrm{H}^2\left(\Delta\mathfrak{g}_e, \mathbb{R}\right) = \mathrm{H}^0\left(\Delta\mathfrak{g}_e, \mathrm{H}^2\left(R\left(\Delta\mathfrak{g}_e\right), \mathbb{R}\right)\right)$$

Without loss of generality, we may suppose the ground field to be the field of complex numbers. Let $S \subset \Delta\mathfrak{g}_e$ a Levi sub-algebra of $\Delta\mathfrak{g}_e$, and let $\mathfrak{h} \subset S$ be a Cartan subalgebra. Let α be a weigh of the S-module $R(\Delta\mathfrak{g}_e)$ and let us denote by $r_\alpha \in R(\Delta\mathfrak{g}_e)$ be a non zero element such that $[H, r_\alpha] = \alpha(H)r_\alpha$, $\forall H \in \mathfrak{h}$, It is a straight consequence of (alla 3) that $R(\Delta\mathfrak{g}_e)$ is spanned by the elements r_α. Thus, every cohomology class $[\theta] \in \mathrm{H}^0\left(\Delta\mathfrak{g}_e, \mathrm{H}^2\left(R\left(\Delta\mathfrak{g}_e\right), \mathbb{R}\right)\right)$ may be represented by a cocycle $\theta \in \mathrm{Hom}\left(R(\Delta\mathfrak{g}_e), \mathbb{R}\right)$ such that $\forall H \in \mathfrak{h}, L_H\theta$ is the differential of a $\phi_H \in \mathrm{Hom}\left(R(\Delta\mathfrak{g}_e), \mathbb{R}\right)$. . So, given two elements $r_\alpha, r_{\alpha'}$ as defined above, one has:

$$(\alpha + \alpha')(H)\theta(r_\alpha, r_{\alpha'}) = \phi_H\left([r_\alpha, r_{\alpha'}]\right).$$

Since $R(\Delta\mathfrak{g}_e)$ is commutative, the last identity implies the following one:

$$(\alpha+\alpha')(H)\theta(r_\alpha, r_{\alpha'}) = 0.$$

Thus we see that $\mathrm{H}^2\left(R(\Delta\mathfrak{g}_e), \mathbb{R}\right) = 0$. So, if ω is a left invariant symplectic form on G, then ω is the differential of a left invariant 1-form θ Thereby, the orbit of θ under the coadjoint action of G is an open set in the dual vector space $\mathfrak{g}^*$. If the ground field $\mathbb{K}$ is the field of complex numbers, then $\mathrm{Ad}^*G(\theta)$ is the complement of an algebraic variety. Indeed, given an element $\eta \in \mathfrak{g}^*$, let x_η, be the element of $\mathfrak{g}$ defined by:

$$i(x_\eta)\omega = \eta.$$

Let $\theta \in \mathfrak{g}^*$,let us define the linear endomorphism $\Phi_\theta$of $\mathfrak{g}^*$ by setting:

$$\Phi_\theta(\eta) = \mathrm{ad}^*(x_\eta\theta).$$

The orbit of θ is an open set in $\mathfrak{g}^*$ if and only if Φ_θ is injective; in this case, $\mathrm{Ad}^*G(\theta)$ is an open and dense orbit. Since we assumed the ground field to be $\mathbb{C}$, it is unique. □

Theorem 2 yields the following statement.

Theorem 6. *Let (n, m) be a pair of positive integers. If $n > 2$, then one has:*

$$H^2\left(\mathfrak{g}_{n,m}, \mathbb{R}\right) = 0$$

In particular, (see [RAI],) those of the $g_{n,m}$ which admit symplectic forms are Frobeniusian.

6. A-algebroids

This section will be devoted to a relevant application of Theorem 4. To begin with, let us recall the following normal forms theorem of Jean-Paul Dufour, [9] (see also [1, 3–5, 16].)

Theorem 7. *DUF2 Let (A, a) be a Lie algebroid of rank n over a smooth m- dimensional manifold M. Let r be the rank of the anchor map a at the point $p \in M$. Then, on an open neighborhood U of p, there exist:*

(1) A basis of local sections of A, say $\{\sigma_1, \ldots, \sigma_r, \tau_1, \ldots, \tau_{n-r}\}$,

(2) A system of local coordinate functions on M, say $\{x_1, \ldots, x_r, y_1, \ldots, y_{m-r}\}$

Such that the next conditions are satisfied for any $j = 1, \ldots, r$; $k, l = 1, \ldots, n-r$:

i) $a(\sigma_j) = \partial x_j$ and $a(\tau_k)(p) = 0$,

ii) $a(\tau_k)$ doesn't depend on $x_1, \ldots, x_r$,

iii) $[\sigma_j, \tau_k] = 0$,

iv) $[\tau_k, \tau_l] = \sum_\nu c^\nu_{kl}\tau_\nu$, the functions c^ν_{kl} depending only on $y_1, \ldots, y_{m-r}$.

In [4], R. Wolak and the author given an another proof of Theorem 7 which works for Koszul-Vinberg algebroids as well. Both proofs in [9] and in [4] are based in differential geometry arguments. The author intends to read the theorem of J-P Dufour from the sheaf theoretic point of view. Really, Theorem 7 is different from the local decomposition of Poisson manifolds, [22]. Let (A, a) be a smooth or analytic algebroid over a smooth or analytic manifold M. Let us denote by $\Gamma(A)$ (resp. $\Gamma(TM)$) the sheaf of local smooth sections of A (resp. TM). Because the anchor map a may have non constant rank, the following exact sequence:

$$\ker(a) \longrightarrow A \longrightarrow TM$$

is not a sequence a vector bundles over M. However, the anchor map induces a sheaf homomorphism:

$$\Gamma(A) \to \Gamma(TM)$$

which is still denoted by a. The kernel of the last sheaf homomorphism is a sub-sheaf of $\Gamma(A)$, denoted $\mathrm{Ker}(a)$. It is not locally trivial. However, $\Gamma(A)$ is a sheaf of $\mathrm{Ker}(a)$-modules. Let $\tau \in \Gamma(A), X \in \Gamma(TM)$. Setting $\tau.X = [a(\tau), X]$ makes $\Gamma(TM)$ inherit the trivial $\mathrm{Ker}(a)$-module structure. So, one points out the following exact sequence of $\mathrm{Ker}(a)$-modules:

$$0 \longrightarrow \mathrm{Ker}(a) \longrightarrow \Gamma(A) \longrightarrow a\left(\Gamma(A)\right) \longrightarrow 0$$

Now, let us point out the following observation. The algebraic counterpart of Theorem 7 is that the cohomology class in $\mathrm{H}^1\left(\mathrm{Ker}(a), \mathrm{Hom}\left(a\left(\Gamma(A)\right), \mathrm{Ker}(a)\right)\right)$ whichis represented by the extension above is zero. So, in some particular cases, that algebraic counterpart of Theorem 7 can be proved directly. In the next, we intend to give such a proof for stratified a-algebroids.Let us consider the Leibniz axiom of (A, a). Then, given two sections s, s' of A and a smooth function f, one has:

$$[s, fs'] = f[s, s'] + (a(s)f)\, s'.$$

It is a straight consequence of the identity above that the bracket of sections of A induces a Lie algebra structure on the vector space:

$$\ker(a)(p) = \mathrm{span}\{\tau_1(p), \ldots, \tau_{n-r}(p)\}$$

Definition 3. A lie albegroid (A, a) over a smooth manifold M is called a-algebroid (resp. non solvable a-algebroid) if $\forall p \in M, ker(a)(p))$ is an affinelike Lie algebra (resp. non solvable affinelike Lie algebra).

Our aim is to show how in the case of non solvable a-algebroids, one can use the vanishing theorems of Section 5 to supply an algebraic proof of Theorem 7. Now, let (A, a) be an a-algebroid over the manifold M; Forall $p \in M$, the bracket of sections of A induces an affinelike Lie algebra structure on $\ker(a)(p)$. Let $r(p) = \mathrm{rank}(a(p))$. We put:

$$r_1(a) = \max_p(r(p)).$$

The singular points of (A, a) are elements of the subset $\Sigma(a)$ which consists of those $p \in M$ where $r(p) < r_1$. Of course $M - \Sigma(a)$ is a open subset of M. Let $p \in M - \Sigma(a)$ and let $V \subset M - \Sigma(a)$ be an connected open neighborhood of p. Then, over the sub-manifold V, we get the following exact sequence of vector bundles:

$$0 \longrightarrow \ker(a) \longrightarrow A \longrightarrow a(A) \longrightarrow 0$$

On one side, the sub-vector bundle $a(A) \subset TV$ defines a regular foliation on V. On the other side, $ker(a)$ is a locally trivial bundle (over V) of non solvable affinelike Lie algebras. Indeed, we know that $\forall p \in V$ one has:

$$\mathrm{H}^2\left(\ker(a)(p), \ker(a)(p)\right) = 0.$$

So, all of the Lie algebras $ker(a)(p)$ are rigid, [2, 18]. Thus, $\forall p, p' \in V$, the Lie algebras $ker(a)(p)$ and $ker(a)(p')$ are isomorphic. Over the open set V, the action of $\mathrm{Ker}(a) = \Gamma(ker(a))$ on $X(V)$ is trivial. Re- stricting ourself to V, we regard $a(A)$ and TV as bundles of $\mathrm{Ker}(a)$-modules. For our purpose, we shall need the following result.

Theorem 8. *If T is a trivial module of a non solvable affinelike Lie algebra $(\mathfrak{g}, e)$, then $Ext^1_{\Delta\mathfrak{g}}(T, \mathfrak{g}) = 0$.*

Proof. Keeping the notations of section 2, we consider the following exact sequence of Δ_g-modules:

$$(5) \qquad 0 \longrightarrow \mathrm{Hom}\left(T, \Delta_{\mathfrak{g}}\right) \longrightarrow \mathrm{Hom}\left(T, \mathfrak{g}\right) \longrightarrow \mathrm{Hom}\left(T, \mathfrak{g}/\Delta\mathfrak{g}\right) \longrightarrow 0$$

Step 1.

Let us show that:

$$\mathrm{H}^1\left(\Delta\mathfrak{g}, \mathrm{Hom}\left(T, \Delta\mathfrak{g}\right)\right) = 0.$$

To do that, we assume again the ground field to be the field of complex numbers. Let us fix a Levi subalgebra $S([\mathfrak{g}_e, \mathfrak{g}_e])$ and a Cartan subalgebra $\mathfrak{h} \subset S$. To $(\mathfrak{h}, S)$ we shall also assign a fixed weights system $\{\alpha, \beta, \dots\}$ (for the S-module $\Delta\mathfrak{g}$). Let $\theta \in \mathrm{Hom}\left(\Delta\mathfrak{g}, \mathrm{Hom}\left(T, \Delta\mathfrak{g}\right)\right)$ be a cocycle of $\Delta\mathfrak{g}$. There exists an element $\xi \in \mathrm{Hom}\left(T, \Delta\mathfrak{g}\right)$ such that $\forall s \in S, \forall t \in T$, one has:

$$\left(\theta(s)\right)(t) = \left(L_s\xi\right)(t) = \left[s, \xi(t)\right].$$

Now, let (α, β) be a couple of weights of the S-module $\Delta\mathfrak{g}$ and let r_α be a non zero element of $\Delta\mathfrak{g}$ whose weight is α. Pour tout $t \in T$ et $H \in \mathfrak{h}$, we have:

$$\left[H, \left(\theta(r_\alpha)\right)(t)\right] \left[r_\alpha, \left(\theta(H)\right)(t)\right] = \left(\theta\left(\left[H, r_\alpha\right]\right)\right)(t).$$

Let us compute the β-component of the identity just set. The calculation yields the following identity:

$$(\beta - \alpha)\,\theta_b(r_\alpha) = \left[r_\alpha, \left(\theta(H)\right)(t)\right]_\beta = \left[r_\alpha, \left[H, \left(\xi(t)\right)_{\beta-\alpha}\right]\right] = (\beta - \alpha)(H)\left[r_\alpha, \xi_{\beta-\alpha}(t)\right]$$

The last identity will hold if and only if $\forall \alpha, \forall t \in T$, one has:

$$\theta(r_\alpha)(t) = \left[r_\alpha, \xi(t)\right].$$

So we have to conclude that:

$$\mathrm{H}^1\left(\Delta\mathfrak{g}, \mathrm{Hom}\left(T, \Delta\mathfrak{g}\right)\right) = 0.$$

Step 2.

It is convenient now to observe that the structure of $\Delta\mathfrak{g}$-module of $\mathrm{Hom}\left(T, \mathfrak{g}, \Delta\mathfrak{g}\right)$ is trivial. Thus, we can identify $\mathrm{H}^1\left(\Delta\mathfrak{g}, \mathrm{Hom}\left(T, \mathfrak{g}/\Delta\mathfrak{g}\right)\right)$ with $\left(\mathrm{H}^1\left(\Delta\mathfrak{g}, \mathbb{K}\right)\right)^{\dim T}$.Since the Lie algebra $\Delta\mathfrak{g}$ is perfect, $\mathrm{H}^1\left(\Delta\mathfrak{g}, \mathbb{K}\right) = 0$. Taking into account both *Step 1* and *Step2* above, we deduce from the exact cohomology sequence given by the exact sequence 5 that $\mathrm{H}^1\left(\Delta\mathfrak{g}, \mathrm{Hom}\left(T, \mathfrak{g}\right)\right) = 0$. □

An algebraic proof of theorem 7

Proof. Let us keep the notations already used above. Let us fix a connected open set $V \subset M - \Sigma(a)$. From the sheaf theoretic point of view, one is interested in the following sheaf cohomology:

$$\mathrm{H}^1\left(\Delta\mathrm{Ker}(a), \Gamma\left(\mathrm{Hom}\left(a(A), \mathrm{Ker}(a)\right)\right)\right).$$

By taking into account Theorem 8, one sees that:

$$\mathrm{H}^1\left(\Delta\mathrm{Ker}(a), \Gamma\left(\mathrm{Hom}\left(a(A), \mathrm{Ker}(a)\right)\right)\right) = 0.$$

In other words, one has:

$$\mathrm{Ext}^1_{\Delta\mathrm{Ker}(a)}\left(\left(a\left(\Gamma(A)\right), \mathrm{Ker}(a)\right)\right) = 0.$$

Thereby, the following exact sequence of $\Delta\Gamma(ker(a))$- modules splits:

$$0 \longrightarrow \mathrm{Ker}(a) \longrightarrow \Gamma(A) \longrightarrow a\left(\Gamma(A)\right) \longrightarrow 0$$

The last assertion means that every point $p \in V$ has an open neighborhood U on which the following claims hold:

(a) It exists a partial basis of local sections of A, say $(\sigma_i)_{i=1\ldots r}$, such that:

$$\forall p \in U,\ \mathrm{rank}\{a(\sigma_1)(p), \ldots, a(\sigma_r)(p)\} = r;$$

(a) It exists a basis of local sections of $\Delta\mathrm{Ker}(a)$, say $(\tau_j)_{j=1\ldots n-r-1}$ satisfying:

$$\left[\sigma_i, \tau_j\right] = 0.$$

To yield the normal forms theorem of [9], let us assume the domain U of $\left(\sigma_i, \tau_j\right)$ to be also a domain of local coordinate functions of V, say $(x_i, y_l)_{i=1\ldots r, l=1\cdots-r}$ such that the connected components of the leaves of $a(A)$ which are included in U are defined by the following systems:

$$y_l = \mathrm{cte},\ l = 1\ldots m-r$$

Thereby, one can choose the local basis (σ_i) such that:

$$a(\sigma_i) = \partial x_i,\ i = 1\ldots r.$$

The last conditions imply that:

$$\left[\sigma_i, \sigma_{i'} = 0\right].$$

Therefore, the property $[]\sigma_i, \tau_l] = 0$ implies that:

$$[\tau_l, \tau_{l'}] = \sum_{l''} c^{l''}_{ll'}(y_1, \ldots, y_{m-r})\tau_{l''}.$$

To end the "algebraic" proof of Theorem 7, let us remind that all of the fibers of $(\mathrm{ker}(a))\,(p), p \in V$, are isomorphic to a fixed non solvable affinelike Lie algebra $(\mathfrak{g}, e)$. So, $\mathrm{Ker}(a)$ may be regarded as a $(\mathfrak{g}, e)$-Current algebra over V, [12]. Now, to end the proof, let us assume that U is also a domain of trivialization of (A, a) as well. So, as vector bundle over U, $\mathrm{ker}(a)$ isomorphic to $Ux(\mathfrak{g}, e)$. Let τ_0 be the section of A defined by:

$$\tau_0(p) = (p, e), \forall p \in U.$$

Now, to every couple $(u, s) \in \Delta_e\mathfrak{g} \times S$, we assign the couple (v, τ) of sections of A which is defined by:

$$v(p) = (p, u)$$
$$\tau(p))(p, s)$$

Actually, one easily sees that for $i = 1\ldots r$, $[\tau_0, \sigma_i] \in \Gamma\left(\mathrm{ker}(a)\right)$. On one hand, the identity:

$$[\tau, [\tau_0, \sigma_i]] = 0$$

shows that $[\tau_0, \alpha_i] \in \Delta\mathrm{Ker}(a)$. On the other hand, the identity:

$$[\tau, [\tau_0, \sigma_i]] = 0$$

shows that $\forall p \in V,\ [\tau_0, \sigma_i](p) \in \mathrm{H}^0(S, \Delta\mathfrak{g})$. Since $(\mathfrak{g}, e)$ is non solvable, one has to conclude that:

$$[\tau_0, \sigma_i] = 0.$$

Thus, the system:

$$\{\tau_l, \sigma_i\}_{l=0\ldots n-r, i=1\ldots r}$$

is a basis of local sections of A satisfying the properties required in Theorem 7.To get the general case, let us assume that the a-algebroid (A, a) is stratified in following sense. The manifold M admits a filtration:

$$M = \Sigma_0 \supset \Sigma_1 \supset \cdots \supset \Sigma_i \supset \ldots$$

with the next property. Let $(A_i, a_i;)$ be the pull back of (A, a) by the inclusion map of Σ_i in M. Then:

(Str 1) $\Sigma_{i=1}$ consists of singular points of (A_i, a_i).

(Str 2) The family $\Sigma_i - \Sigma_{i+1}$ is locally finite.

Claim *(Str 1)* implies that $a_i(A_i)$ defines a regular foliation on $\Sigma_i - \Sigma_{i+1}$. Our algebraic proof of Theorem 7 works for the restriction $\Sigma_i - \Sigma_{i+1}$ of (A_i, a_i). Actually all of the Σ_i are closed sub-manifolds of M. Let d_i be the dimension of σ_i. Every point of M has an open neighborood U which is the domaine of local coordinate functions of M, say $(x_1, \ldots, x_m)$ such that the connected components of $U \cap \Sigma_i$ are defined by the system

$$x_{d_i+1} = \text{cte}, \ldots, x_m = \text{cte}.$$

The last assertion yields Theorem 7 for stratified a-algebroid. □

Let us make the following observations. Firstly, a Lie algebroid (A, a) over the manifold M is called s — algebroid if $\forall p \in M, \ker(a)(p)$ is a semi-simple Lie algebra. Now, we observe that all of the cohomology arguments which are used in the alge- braic proof of Theorem 7 work for s-algebroids. Secondly, R. Wolak and the author have proved the analogue to Theorem 7 for Koszul-Vinberg algebroids, [4]. Our analytic proof of normal forms theorem for Koszul-Vinberg algebroids might have its algebraic counterpart in KV-cohomolgy [2, 3, 17].

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