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Abstract In this paper, we show how a reliable global Branch and Bound optimization method based on interval arithmetic can be used efficiently to numerically prove a conjecture in geometry about how to cover a rectangle by 6 circles of equal radius.

Keywords: Reliable Global Optimization, Mathematical Programming, Numerical proof, Covering problem

1. Introduction

We adress a covering problem, where, given a rectangle and a fixed number n of circles having all the same radius, one wants to find the minimal value of the radius as well as to decide the position of the circles in such a way that the rectangle is completely covered. Circles are allowed to overlap. Problems of this kind are clearly optimization problems. They are usually addressed via geometrical approaches and sometimes stochastic optimization algorithms [3]. Mathematical programming, and exact global optimization, are more rarely employed.

Let us consider the case of $n = 6$ circles, and rectangle side lengths respectively 1 and $a \geq 1$. In a seminal paper [3], Melissen and Schuur presented the different configurations of the six circles, i.e. the different ways the circles must be placed to optimally cover the rectangle, according to the (increasing) value of a. For two such configurations, they could not obtain an analytical expression of the minimal radius, although for one of them, named *configuration b)*, they obtained a mathematical expression of the relation between the minimal radius r and a. These results were actually not proved to be optimal. In [1], it has been proved that the circle configurations are actually optimal configurations, and that the relation between r and a for configuration b) holds at the optimal solution. From this relation, an analytical expression of the minimal radius was obtained by using `Maple` (a symbolic computation software). This expression is highly nonlinear, and involves a complex number. Thus, one can seek alternative ways for computing optimal solutions, especially in practical contextes. We propose to use Mathematical Programming to formulate the covering problem, and solve the problem by exact global optimization methods. We propose in particular a mathematical programming formulation whose solution provides a numerical proof that the relation of Melissen and Schuur is optimal. We solve such a problem by interval Branch-and-Bound, that allows us to compute exact solutions which are reliable to numerical errors.

2. A Mathematical Programming formulation

Let us consider the case of circle configuration b), corresponding to a varying in the range $[2.92, 3.12]$, and displayed in Figure 1.

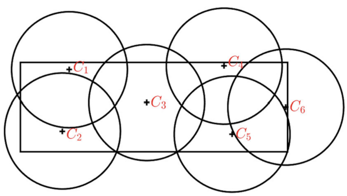

Figure 1. Covering a rectangle by 6 circles: configuration b), [3].

The mathematical relation between r and a (holding for $r \geq \frac{1}{2}$) found by Melissen and Schuur [3], and proved in [1] to occur at the optimal solution for lengths a in the interval $[2.92, 3.12]$, reads:

$$a = 2\sqrt{4r^2 - 1} + 2r + \sqrt{4r^2 - \frac{1}{4}} \quad (1)$$

On the other hand, the optimal value of r can be (numerically) computed as the solution of an optimization problem whose decision variables are r together with the coordinates of the circle centers, where the objective function to be minimized is simply r, and the constraints ensure that the rectangle is entirely covered. Given the set of vertices of the rectangle $V = \{(0,0),(1,0),(1,1),(0,1)\}$, the set of circle centers $C = \{(x_i, y_i), i = 1,..,6\}$, and the set of points representing the intersection of pairs of circles inside the interior of the rectangle $C^{in} = \{(x_{jk}, y_{jk}) : \mathrm{circle_j} \cap \mathrm{circle_k} \neq \emptyset\}$ this optimization problem, that we refer to as (P_b), can be formulated as follows:

$$
\begin{aligned}
\min_{x_i, y_i, r} \quad & r \\
s.t. \quad & \\
& \mathcal{G}^v(c, v, r) \leq 0 \qquad v \in V,\ c \in C_v \subset C & (2)\\
& \mathcal{G}^s(c^j, c^k, r) \leq 0 \qquad c^j, c^k \in C_s \subset C & (3)\\
& \mathcal{G}^{in}(c^{in}, c, r) \leq 0 \qquad c^{in} \in C^{in},\ c \in C_i \subset C & (4)\\
& 0 \leq r \leq 1 \\
& 0 \leq x_i \leq a, \qquad 0 \leq y_i \leq 1 \quad \forall i \in \{1, \ldots, 6\}
\end{aligned}
$$

where (2), (3) and (4) are three sets of constraints ensuring respectively the covering of the vertices of the rectangle, of its sides and of its interior. For each value of a in the interval range [2.92, 3.12] characterizing configuration b), solving the problem above gives optimal values of r and of the coordinates of circle centers. Building on this problem, we introduce an additional mathematical programming formulation to the aim of numerically proving the Melissen and Schuur's relation (1). In this formulation, decision variables are $a \in [2.92, 3.12]$, the coordinates (x_i, y_i) of the circle centers, r, and r^* representing the radius that satisfies equation (1). The constraints are: the constraints of (P_b) (with respect to variables x_i, y_i and r), equation (1), a constraint imposing that the difference between r and r^* is nonzero, and bounds on the variables. The objective, to be maximized, is the gap between r^* and r. We impose that this gap is nonzero (with an ϵ tolerance), so that, proving that the problem at hand is infeasable, we prove that no r different from r^* can be an optimal solution of the problem. This implies that equation (1) is satisfied at the optimum. The formulation is the following:

$$
\begin{aligned}
\max_{a, x_i, y_i, r^*, r} \quad & r^* - r \\
s.t. \quad & \\
& (2), (3), (4) & (5)\\
& a = 2\sqrt{4r^{*2} - 1} + 2r^* + \sqrt{4r^{*2} - \tfrac{1}{4}} \\
& r + \epsilon \leq r^*
\end{aligned}
$$

3. Solving the problem by interval computation-based Global Optimization

We address the solution of the problem presented in the previous section by exact global optimization. On a computer system, calculations between real numbers are based on floating-point representations and operations. This introduces numerical errors and therefore, using numerical optimization software, it is generally not possible to certify the results computed for problems such as (5). Nevertheless, by using interval arithmetic introduced by Moore [4] and extended by Kearfott [2] in order to rigorously solve global optimization problems, it becomes possible to reliably solve problem (5). We use the `IBBA` [5] solver, which implements a Branch-and-Bound based on interval computations, affine arithmetic, and reliable computation of bounds. `IBBA` proves in less than 3h that problem (5) has no solution within a tolerance $\epsilon = 10^{-6}$. Thus, for all $a \in [2.92, 3.12]$, the conjecture of Melissen and Schuur is proven to be ϵ-optimal (with $\epsilon = 10^{-6}$).

4. Summary

In this paper, we show that equation (1) of Melissen and Shuur [3] can be proved numerically within a given tolerance 10^{-6} by associating a mathematical programming model with a reliable global optimization solver.

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